

ON GENERALIZED STATISTICALLY CONVERGENT SEQUENCES

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This article introduces $\bar{c}(p)$ and $\bar{c}_0(p)$ for $p = (p_k)$ a sequence of non-negative real numbers. This generalizes the spaces of statistically convergent and statistically null sequences. Some properties of these spaces are studied.

Key Words : Density; Separable; Statistically Convergent; Statistically Null; Paranorm

1. INTRODUCTION

In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast², Schoenberg¹⁴ and Zygmund¹⁹ independently. Later on it was studied and linked with summability by Fridy^{3, 4}, Connor¹, Šalát¹³, Kolk⁵, Rath and Tripathy^{11 & 12}, Tripathy^{16, 17 & 18} and many others. The idea depends on the density of subsets of the set N of natural numbers. A subset A of N is said to have *density* $\delta(A)$ if

$$\delta(A) = \lim_{n \rightarrow \infty} \frac{1}{n} \sum_{k=1}^n \chi_A(k) \text{ exists.}$$

where X_A is the characteristic function of A .

Clearly finite subsets of N have zero natural density and $\delta(A^c) = \delta(N - A) = 1 - \delta(A)$. For (x_k) and (y_k) two sequences, we say that $x_k = y_k$ for *almost all* k (in short a.a.k) if $\delta(\{k \in N : x_k \neq y_k\}) = 0$.

A sequence (x_k) is said to be *statistically convergent* to L if for any $\varepsilon > 0$, $\delta(\{k \in N : |x_k - L| \geq \varepsilon\}) = 0$ and we write $x_k \xrightarrow{stat} L$.

A sequence (x_k) is said to be a *statistically Cauchy sequence* provided that for every $\varepsilon > 0$ there exists a number n_0 such that $|x_k - x_{n_0}| < \varepsilon$ a.a.k.

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A subsequence (x_{k_n}) of (x_n) is said to be a *thin* subsequence of (x_n) if $\delta(\{k_n : n \in N\}) = 0$. It is said to be *non-thin* subsequence if either $\{k_n : n \in N\}$ has non-zero density or fails to have density.

2. DEFINITIONS AND BACKGROUND

Throughout the article $s, l_\infty, c, c_0, l_1, l_p, \bar{c}, \bar{c}_0, m, m_0$ will denote the spaces of *all, bounded, convergent, null, absolutely summable, p-absolutely summable, statistically convergent, statistically null, bounded statistically convergent* and *bounded statistically null* sequences respectively. Further $p = (p_k)$ will denote a sequence of strictly positive numbers (not necessarily bounded in general).

The concept of generalized sequences was studied by Nakano¹⁰ and Simons¹⁵ at the initial stage. Later on it was studied by Maddox^{8 & 9}, Lascarides^{6 & 7} and many others. Generalizing l_∞, c, c_0, l_p people have studied $l_\infty(p), c(p), c_0(p), l(p)$ etc. and have characterized some matrix classes.

Let $p = (p_k)$ be a real sequence and $H = \sup_k p_k < \infty$.

Then we have the following well known inequality

$$|a_k + b_k|^{p_k} \leq C(|a_k|^{p_k} + |b_k|^{p_k}), \quad \dots (2.1)$$

where $C = \max(1, 2^{H-1})$.

On generalizing the sequence spaces \bar{c} and \bar{c}_0 we define the following spaces

$$\bar{c}(p) = \{(x_k) \in S : |x_k - L|^{p_k} \xrightarrow{stat} 0, \text{ as } k \rightarrow \infty \text{ for some } L\}$$

$$\bar{c}_0(p) = \{(x_k) \in S : |x_k|^{p_k} \xrightarrow{stat} 0, \text{ as } k \rightarrow \infty\}.$$

With the help of inequality (2.1), one can easily see that $m(p) = \bar{c}(p) \cap l_\infty(p)$ is an additive group, the addition being co-ordinatewise addition. From the existing results, we find that $l_\infty(p), c(p), c_0(p), l(p)$ are linear spaces if and only if $(p_k) \in l_\infty$. It can be easily verified that $m(p)$ and $m_0(p)$ are linear spaces if and only if $(p_k) \in l_\infty$. The spaces $l_\infty(p), c(p), c_0(p)$ are paranormed by

$$g(x) = \sup_k |x_k|^{p_k/M}$$

and $l(p)$ by

$$f(x) = \left(\sum_k |x_k|^{p_k} \right)^{1/M},$$

where $M = \max(1, H)$. It is clear that the spaces $m(p)$ and $m_0(p) = \bar{c}_0(p) \cap l_\infty(p)$ are also paranormed by g .

The following results will be used for establishing the results of this article.

Lemma 1 (Fridy³, Theorem 1) — The following statements are equivalent.

- (i) (x_k) is a statistically convergent sequence;
- (ii) (x_k) is a statistically Cauchy sequence;
- (iii) (x_k) is a sequence for which there is a convergent sequence (y_k) such that $x_k = y_k$ a.a.k.

Lemma 2 (Šalát¹³, lemma 1.1) — A sequence (x_k) statistically converges to L if and only if there exists such a set $K = \{k_1 < k_2 < \dots < k_n < \dots\} \subset N$ that $\delta(K) = 1$ and $\lim_{n \rightarrow \infty} x_{k_n} = L$.

Lemma 3 (Connor¹, Theorem 2.3) — If $x \in s$ is statistically convergent to L , then there is a convergent sequence y and a statistically null sequence z such that y is convergent to L , $x = y + z$ and $\delta(\{k \in N : z_k \neq 0\}) = 0$. Moreover, if x is bounded then z is bounded and $\|z\|_\infty \leq \|x\|_\infty + |L|$.

3. MAIN RESULTS

In this section we prove some results relating to $m(p)$ and $m_0(p)$.

The following results is the decomposition theorem for $m(p)$. For $p_k = 1$, for all $k \in N$ we have the decomposition theorem earlier proved by Fridy³, Šalát¹³ and Connor¹ for m and m_0 .

Theorem 1 — Let $0 < \inf p_k \leq p_k \leq \sup p_k < \infty$, then the following are equivalent.

- (a) $(x_k) \in m(p)$
- (b) $(x_k - L) \in m_0(p)$ for some L
- (c) there exists $(y_k) \in c(p)$ such that $x_k = y_k$ for a.a.k
- (d) there exists a subset $K = \{k_1, k_2, \dots\}$ of N such that $\delta(K) = 1$ and

$$\lim_{n \rightarrow \infty} |x_{k_n} - L|^{p_{k_n}} = 0$$

- (e) there exists sequences (y_k) and (z_k) such that $x_k = y_k + z_k$ for all $k \in N$ and $(y_k) \in c(p), (z_k) \in m_0(p)$.

PROOF : The equivalence of (a) and (b) is clear from the definitions.

Let $(x_k) \in m(p)$. Then there exists L such that $|x_k - L|^{p_k} \xrightarrow{stat} 0$. Let $z_k = |x_k - L|^{p_k}$, then $(z_k) \in m_0(p)$. Now the equivalence of (a) and (c) follows from lemma 1, that of (a) and (d) from Lemma 2 and that of (a) and (e) from lemma 3.

Remark : We have the condition $0 < \inf p_k \leq p_k \leq \sup p_k < \infty$ on (p_k) in the above theorem. The condition $\sup p_k < \infty$ is necessary for the linearity of the space $c(p)$ (see Maddox⁹). We have established the equivalence of the relations involving $c(p)$. Hence, $\sup p_k < \infty$ is necessary. Further

Simons¹⁵ proved that $l_\infty(p)$ is paranormed by $g(x) = \sup_k |x_k| \frac{p_k}{M}$ only if $\inf p_k > 0$. So from the definition of $m(p)$, $\inf p_k > 0$ is necessary.

Theorem 2 — $m(p)$ is a closed subspace of $l_\infty(p)$.

PROOF : Let $(x^{(n)})$ be a Cauchy sequence in $m(p)$ such that $x^{(n)} \rightarrow x$ in $l_\infty(p)$.

We show that $x \in m(p)$. Since $x^{(n)} \in m(p)$, so there exists a_n such that

$$|x_k^{(n)} - a_n|^{p_k} \xrightarrow{stat} 0, \text{ as } k \rightarrow \infty.$$

To show that

(i) (a_n) converges to a

and (ii) $|x_k - a|^{p_k} \xrightarrow{stat} 0$.

(i) Since $(x^{(n)})$ is a convergent sequence of elements, so for a given $\epsilon > 0$, there exists $n_0 \in N$ such that

$$\sup_k |x_k^{(n)} - x_k^{(m)}| \frac{p_k}{M} < \frac{\epsilon}{3} \text{ for all } m, n \geq n_0.$$

Again given $\epsilon > 0$, we have

$$\delta(A_m) = \delta \left(\left\{ k \in N : |x_k^{(m)} - a_m|^{p_k} < \left(\frac{\epsilon}{3} \right)^M \right\} \right) = 1$$

and
$$\delta(A_n) = \delta \left(\left\{ k \in N : |x_k^{(n)} - a_n|^{p_k} < \left(\frac{\epsilon}{3} \right)^M \right\} \right) = 1.$$

Let $A = A_m \cap A_n$, then $\delta(A) = 1$. We choose $k \in A$. Then for each $m, n \geq n_0$, we have

$$|a_m - a_n| \leq |a_m - x_k^{(m)}| + |x_k^{(m)} - x_k^{(n)}| + |x_k^{(n)} - a_n|.$$

Thus (a_n) is a Cauchy sequence of scalars, so there exists a scalar a such that $\lim_{k \rightarrow \infty} a_k = a$.

(ii) Let $\eta > 0$, we show that

$$\delta(F) = \delta (\{k \in N : |x_k - a|^{p_k} < \eta\}) = 1.$$

Since $x^{(n)} \rightarrow x$, there exists $q \in N$ such that

$$\| (x_k^{(q)}) - (x_k) \| < \left(\frac{\eta}{3} \right)^{1/M} \quad \dots (3.1)$$

The number q can be chosen in such a way that together with (3.1) we have

$$| a_q - a |^{p_k} < \frac{\eta}{3} \quad \dots (3.2)$$

Again since $| x_k^q - a_q |^{p_k} \xrightarrow{stat} 0$, we have a subset B of N such that $\delta(B) = 1$,

where
$$B = \left\{ k \in N : | x_k^{(q)} - a_q |^{p_k} < \frac{\eta}{3} \right\}.$$

Therefore, for each $k \in B$ we have

$$\begin{aligned} | x_k - a |^{p_k} &\leq C^2 | x_k - x_k^{(q)} |^{p_k} + C^2 | x_k^{(q)} - a_q |^{p_k} + C | a_q - a |^{p_k} \\ &< C^2 \left(\frac{\eta}{3} \right) + C^2 \left(\frac{\eta}{3} \right) + C \left(\frac{\eta}{3} \right) = \eta' \text{ (say).} \end{aligned}$$

This completes the proof.

Theorem 3 — For two sequences (p_k) and (q_k) we have $m_0(p) \supseteq m_0(q)$ if and only if $\liminf_{k \in K} (p_k/q_k) > 0$, where $K \subseteq N$ such that $\delta(K) = 1$.

PROOF : Let $\liminf_{k \in K} \left(\frac{p_k}{q_k} \right) > 0 \quad \dots (3.3)$

and $(x_k) \in m_0(q)$. Then there exists $\alpha > 0$ such that $p_k > \alpha q_k$ for all sufficiently large $k \in K$. Also

$| x_k |^{q_k} \xrightarrow{stat} 0$ since $(x_k) \in m_0(q)$. Hence for $\epsilon > 0$, we have $\delta(L) = 1$, where $L = \{ k \in N : | x_k |^{q_k} < \epsilon \}$.

Let $J = K \cap L$. Then $\delta(J) = 1$. Also for all sufficiently large $k \in J$, we have

$$\begin{aligned} | x_k |^{p_k} &\leq (| x_k |^{q_k})^\alpha \\ \Rightarrow (x_k) &\in m_0(p). \end{aligned}$$

Conversely, let $m_0(q) \subseteq m_0(p)$ but there exists no $K \subset N$ with $\delta(K) = 1$ such that (3.3) holds. Then we get $k_1 < k_2 < k_3 < \dots$ with $\delta(\{k_i : i \in N\}) \neq 0$ such that $i p_{k_i} < q_{k_i}$. Let us define the sequence (x_k) as follows :-

$$x_k = \begin{cases} (1/i)^{\frac{1}{q_{k_i}}}, & \text{if } k = k_i \\ 0, & \text{otherwise.} \end{cases}$$

Then $(x_k) \in m_0(q)$. But we have

$$(x_{k_i})^{p_{k_i}} > \exp \left(- \frac{\log i}{i} \right)$$

This contradicts the fact that $(x_k) \in m_0(p)$. This completes the proof.

From the above result we have the following.

Corollary — For (p_k) and (q_k) two non-negative sequence of reals, we have $m_0(p) = m_0(q)$ if and only if there exists a subset K of N with $\delta(K) = 1$ such that

$$\liminf_{k \in K} \left(\frac{p_k}{q_k} \right) > 0 \text{ and } \liminf_{k \in K} \left(\frac{q_k}{p_k} \right) > 0.$$

We have the following result which is an easy consequence of $c(p)$ and so we omit the proof.

Theorem 4 — Let $h = \inf_k p_k$ then the following are equivalent :

(i) $H < \infty$ and $h > 0$

(ii) $m(p) = m$.

4. PROPERTIES OF \bar{c}

The proof of the following lemma is a routine work.

Lemma — Let $K = \{m_1, m_2, \dots\}$ be an infinite subset of N such that $\delta(K) = 0$.

Let $L = \{(x_k) : x_k = 0 \text{ or } 1 \text{ for } k = m_i, i \in N \text{ and } x_k = 0 \text{ otherwise}\}$. Then L is uncountable.

The proof of the following result is obvious in view of the above lemma.

Theorem 5 — m is not separable.

REFERENCES

1. J. S. Connor, *Analysis*, **8** (1988) 47-63.
2. H. Fast, *Colloq. Math.*, **2** (1951) 241-44.
3. J. A. Fridy, *Analysis* **5** (1985) 301-13.
4. J. A. Fridy, *Proc. Amer. Math. Soc.*, **118**(4) (1993) 1187-92.
5. E. Kole, *Analysis*, **13** (1993) 77-83.
6. C. G. Lascarides, *Pacific Jour. Math.*, **38**(2) (1971) 487-500.
7. C. G. Lascarides, *Indian Jour. Math.*, **25** no. 1 (1983) 41-52.
8. I. J. Maddox, *Quart. J. Math. (Oxford 2nd ser)* **18** no. 72 (1976) 345-55.
9. I. J. Maddox, *Proc. Camb. Phil. Soc.*, **64** (1968) 335-40.
10. H. Nakano, *Proc. Japan Acad.*, **27** (1951) 508-12.
11. D. Rath and B. C. Tripathy, *Indian J. pure appl. Math.* **25** (4) (1994) 381-86.
12. D. Rath and B. C. Tripathy, *Indian J. pure and appl. Math.* **27** (2) (1996), 197-206.
13. T. Šalát, *Math. Slovaca*, **30** (1980) 139-50.
14. I. J. Schoenberg, *Amer. math. Mon.* **66** (1959) 361-75.
15. S. Simons, *Proc. London math. Soc.*, **15**(3) (1965) 422-36.
16. B. C. Tripathy, *J. math. Anal. Appl.* **206** (1997) 448-50.
17. B. C. Tripathy, *J. Anal.* **6** (1998) 27-31.
18. B. C. Tripathy, *Bull. Calcutta math. Soc.* **90** (1998) 259-62.
19. A. Zygmund, *Trigonometric Series. Vol II.* Cambridge University Press, 1993.