SOME FORMULAE OF SPHERICAL ASTRONOMY OBTAINED BY TENSOR METHOD.

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1. Introduction.

The use of tensor methods in obtaining the effect of geocentric parallax on the declination and right ascension of the moon or a planet is already known. (Jeffreys, 1950). The object of this paper is to extend the method of tensors to obtain expressions for the changes in latitude and longitude, right ascension and declination of a star due to (i) annual parallax, and (ii) aberration.

2. Annual Parallax.

Take the origin at the centre of the earth, axis of x towards the first point of Aries, axis of y towards the point on the ecliptic which is 90 degrees ahead of ♈ and the axis of z towards the pole of the ecliptic. Let the co-ordinates of the star be $x_i = r'l_i$, and those of the sun be $\xi_i = a\lambda_i$. Let r be the distance between the sun and the star, then we have

$$r^2 = (x_i-\xi_i)^2 = r'^2-2r'al_i\lambda_i+a^2,$$

or $$r = r'-al_i\lambda_i, \text{ to the first order of } \frac{a}{r'}.$$

Thus $$r' = r+al_k\lambda_k, \text{ changing the dummy affix } i \text{ to } k.$$

Let $\bar{l}_i$ be the direction cosines of the line joining the centre of the sun to the centre of the star. Then

$$\bar{l}_i = \frac{x_i-\xi_i}{r} = \frac{r'l_i-a\lambda_i}{r} = \frac{(r+al_k\lambda_k)l_i-a\lambda_i}{r}$$

or $$\bar{l}_i = l_i+\pi(l_k\lambda_kl_i-\lambda_i) \qquad \ldots \qquad (2)$$

where $\pi\left(\equiv\frac{a}{r}\right)$ is the annual parallax of the star.

Now the direction cosines are given in terms of the angular co-ordinates by

$$\left.\begin{array}{l} l_1 = \cos\beta\cos\lambda, \quad l_2 = \cos\beta\sin\lambda, \quad l_3 = \sin\beta, \\ \lambda_1 = \cos\odot, \quad \lambda_2 = \sin\odot, \quad \lambda_3 = 0 \end{array}\right\} \qquad \ldots \qquad (3)$$

where $\odot$ is the longitude of the sun and λ, β are respectively the longitude and latitude of the star.

Also $$l_k\lambda_k = \cos\beta\cos(\odot-\lambda) \qquad \ldots \qquad (4)$$

For the parallax in latitude, $\Delta\beta$,

$$\cos\beta\,.\,\Delta\beta = l_3-\bar{l}_3 = \pi(\lambda_3-l_k\lambda_kl_3) = -\pi\cos\beta\cos(\odot-\lambda)\sin\beta$$

so that, $$\Delta\beta = -\pi\sin\beta\cos(\odot-\lambda) \qquad \ldots \qquad (5)$$

Again, $\tan\lambda = \frac{l_2}{l_1}$, $\therefore \log\tan\lambda = \log l_2 - \log l_1$

or $$\frac{\sec^2\lambda}{\tan\lambda}\,\Delta\lambda = \frac{l_2-\bar{l}_2}{l_2} - \frac{l_1-\bar{l}_1}{l_1} = \pi\left(\frac{\lambda_2}{l_2}-\frac{\lambda_1}{l_1}\right)$$
$$= \frac{\pi}{\cos\beta}\left(\frac{\sin\odot}{\sin\lambda}-\frac{\cos\odot}{\cos\lambda}\right) = \frac{\pi}{\cos\beta\sin\lambda\cos\lambda}\sin(\odot-\lambda).$$

Hence $$\Delta\lambda = \pi\sec\beta\sin(\odot-\lambda). \quad \ldots \quad (6)$$

COR. Expressing λ and β in terms of the right ascension α, the declination δ and the obliquity of the ecliptic ϵ, we have (Smart, 1931)

$$l_1 = \cos\delta\cos\alpha, \quad l_2 = \sin\delta\sin\epsilon + \cos\delta\cos\epsilon\sin\alpha,$$
$$l_3 = \sin\delta\cos\epsilon - \cos\delta\sin\epsilon\sin\alpha \quad \ldots \quad (7)$$

and $$l_k\lambda_k = \cos\alpha\cos\delta\cos\odot + \sin\delta\sin\epsilon\sin\odot + \sin\alpha\cos\delta\cos\epsilon\sin\odot \quad \ldots \quad (8)$$

From (7) we have

$$\sin\delta = l_3\cos\epsilon + l_2\sin\epsilon \quad \ldots \quad (9)$$

$$\therefore \cos\delta\,.\,\Delta\delta = \cos\epsilon\,.\,(l_3-\bar{l}_3)+\sin\epsilon\,.\,(l_2-\bar{l}_2)$$
$$= \pi\cos\epsilon(\lambda_3 - l_k\lambda_k l_3) + \pi\sin\epsilon\,(\lambda_2 - l_k\lambda_k l_2)$$
$$= \pi\sin\epsilon\sin\odot - \pi\sin\delta(\cos\alpha\cos\delta\cos\odot + \sin\delta\sin\epsilon\sin\odot$$
$$+\sin\alpha\cos\delta\cos\epsilon\sin\odot)$$

Hence $$\Delta\delta = \pi(\cos\delta\sin\epsilon\sin\odot - \cos\alpha\sin\delta\cos\odot - \sin\alpha\sin\delta\cos\epsilon\sin\odot) \quad \ldots \quad (10)$$

Again, from (7) $\cos\alpha\cos\delta = l_1$,

$$\sin\alpha\cos\delta = \frac{l_2 - \sin\delta\sin\epsilon}{\cos\epsilon} = \frac{l_2-(l_3\cos\epsilon + l_2\sin\epsilon)\sin\epsilon}{\cos\epsilon} = l_2\cos\epsilon - l_3\sin\epsilon \quad \ldots \quad (11)$$

$$\therefore \tan\alpha = \frac{l_2\cos\epsilon - l_3\sin\epsilon}{l_1}, \quad \ldots \quad (12)$$

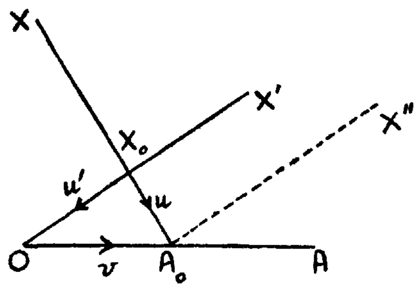
$$\log\tan\alpha = \log(l_2\cos\epsilon - l_3\sin\epsilon) - \log l_1$$

$$\therefore \frac{\sec^2\alpha}{\tan\alpha}\cdot\Delta\alpha = \frac{(l_2-\bar{l}_2)\cos\epsilon - (l_3-\bar{l}_3)\sin\epsilon}{l_2\cos\epsilon - l_3\sin\epsilon} - \frac{l_1-\bar{l}_1}{l_1}$$
$$= \frac{\pi(\lambda_2\cos\epsilon - \lambda_3\sin\epsilon) - \pi l_k\lambda_k(l_2\cos\epsilon - l_3\sin\epsilon)}{l_2\cos\epsilon - l_3\sin\epsilon} - \frac{\pi(\lambda_1 - l_k\lambda_k l_1)}{l_1}$$
$$= \frac{\pi\cos\epsilon\sin\odot}{\sin\alpha\cos\delta} - \frac{\pi\cos\odot}{\cos\alpha\cos\delta}.$$

Hence $$\Delta\alpha = \pi\sec\delta(\cos\alpha\cos\epsilon\sin\odot - \sin\alpha\cos\odot) \quad \ldots \quad (13)$$

3. ABERRATION.

Let X be the true position of a star, A the apex towards which the observer's motion is directed. Let v be the velocity of the observer and u the velocity of light. Along OA mark off length OA_0 to represent v and along XA_0 mark off the length X_0A_0 to represent u, then X_0O represents the velocity of light relative to the observer, say u'. Draw A_0X'' parallel to OX'. Then A_0X'' (or OX') is the apparent direction of the star while A_0X is the true direction.



Let the longitude and latitude of the star be λ and β respectively and let $\odot$ be the longitude of the sun, so that the longitude of the apex is $\odot - 90°$.

Let the co-ordinate axes be the same as in § 2. Let the co-ordinates of X_0 be $x_i = u'l_i$, and those of A_0 be $\xi_i = v\lambda_i$.

Then we have $$u^2 = (x_i - \xi_i)^2 = u'^2 - 2u'vl_k\lambda_k + v^2$$

or $$u = u' - vl_k\lambda_k \quad \ldots \quad (14)$$

to the first order of $\frac{v}{u'}$. $\therefore\ u' = u + vl_k\lambda_k \quad \ldots \quad (14')$

Then the direction cosines of the line A_0X_0 are $\bar{l}_i$ where

$$\bar{l}_i = \frac{x_i - \xi_i}{u} = \frac{u'l_i - v\lambda_i}{u} = \frac{(u + vl_k\lambda_k)l_i - v\lambda_i}{u} = l_i + k(l_k\lambda_k - \lambda_i) \quad \ldots \quad (15)$$

where $k \equiv \frac{v}{u}$, the constant of aberration.

Again the values of l_i are given by (3) or (7) while

$$\lambda_1 = \sin\odot,\ \lambda_2 = -\cos\odot,\ \lambda_3 = 0 \quad \ldots \quad (16)$$

$$\therefore\ l_k\lambda_k = \cos\beta\sin(\odot - \lambda).$$

Let $\Delta\beta$ and $\Delta\lambda$ be the changes in latitude and longitude due to aberration, then we have $\cos\beta\,.\,\Delta\beta = l_3 - \bar{l}_3 = k(\lambda_3 - l_k\lambda_k l_3) = -k\cos\beta\sin\beta\sin(\odot - \lambda)$

or $$\Delta\beta = -k\sin\beta\sin(\odot - \lambda) \quad \ldots \quad (17)$$

Also since $\tan\lambda = \frac{l_2}{l_1}$, we have, $\frac{\sec^2\lambda}{\tan\lambda}\Delta\lambda = \frac{l_2 - \bar{l}_2}{l_2} - \frac{l_1 - \bar{l}_1}{l_1}$

$$= -\frac{k}{\cos\beta\sin\lambda\cos\lambda}\cos(\odot - \lambda)$$

$$\therefore\ \Delta\lambda = -k\sec\beta\cos(\odot - \lambda) \quad \ldots \quad (18)$$

Again, using (9) we have $\cos\delta\,.\,\Delta\delta = \cos\epsilon\cdot(l_3 - \bar{l}_3) + \sin\epsilon\cdot(l_2 - \bar{l}_2)$

$$\therefore\ \Delta\delta = -k\cos\odot\cos\epsilon(\tan\epsilon\cos\delta - \sin\alpha\sin\delta) - k\cos\alpha\sin\delta\sin\odot, \quad \text{(from (15) and (16))} \quad \ldots \quad (19)$$

Also from (12) proceeding exactly as in §2 and using (15) and (16) we get

$$\Delta\alpha = -k\sec\delta(\cos\alpha\cos\odot\cos\epsilon + \sin\alpha\sin\odot) \quad \ldots \quad (20)$$

References.

H. & B. S. Jeffreys, (1950). Methods of Mathematical Physics, 111.
Smart (1931). Spherical Astronomy, 40.