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SIMULTANEOUS TRIPLE INTEGRAL EQUATIONS INVOLVING *G*-FUNCTIONS OF TWO VARIABLES

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A formal solution of simultaneous triple integral equations involving *G*-functions of two variables as kernel by the method of fractional integration operators is obtained.

1. INTRODUCTION

Recently Masood and Srivastava (1970) have obtained the solution of dual integral equations involving *G*-functions of two variables by using fractional integral operators. In this paper we have obtained the solution of simultaneous triple integral equations involving *G*-functions of two variables. The *G*-function of two variables due to Agrawal (1965) is defined as

$$G^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,(t+\nu_1,t+\nu_2),s,(q+m_1,q+m_2)}\left[\begin{array}{c|l} & (\epsilon_{p+n}) \\ x & (\gamma_{t+\nu_1});\ (\gamma'_{t+\nu_2}) \\ & (\delta_s) \\ y & (\beta_{q+m_1});\ (\beta'_{q+m_2}) \end{array}\right] \qquad \ldots(1.1)$$

$$= \frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty} \frac{\prod_{j=1}^{m_1}\Gamma(\beta_j-\xi)\prod_{j=1}^{\nu_1}\Gamma(\gamma_j+\xi)\prod_{j=1}^{m_2}\Gamma(\beta'_j-\eta)\prod_{j=1}^{\nu_2}\Gamma(\gamma'_j+\eta)}{\prod_{j=1}^{q}\Gamma(1-\beta_{m_1+j}+\xi)\prod_{j=1}^{t}\Gamma(1-\gamma_{\nu_1+j}-\xi)\prod_{j=1}^{q}\Gamma(1-\beta'_{m_2+j}+\eta)}$$

$$\times \frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j+\xi+\eta)\,x^{\xi}y^{\eta}}{\prod_{j=1}^{t}\Gamma(1-\gamma'_{\nu_2+j}-\eta)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j}-\xi-\eta)\prod_{j=1}^{s}\Gamma(\delta_j+\xi+\eta)}\,d\xi\,d\eta \qquad \ldots(1.2)$$

where $\quad 0 \leqslant m_1 \leqslant q,\ 0 \leqslant m_2 \leqslant q,\ 0 \leqslant \nu_1 \leqslant t,\ \ 0 \leqslant \nu_2 \leqslant t,\ 0 \leqslant n \leqslant p$

and $\quad (a_r) \equiv a_1, a_2, \ldots, a_r;\ (a_{r+1}, s) \equiv a_{r+1}, a_{r+2}, \ldots, a_s.$

The sequence of parameters (β_{m_1}), (β'_{m_2}), (γ_{ν_1}), (γ'_{ν_2}) and (ϵ_n) are such that none of the poles of the integrand coincide. The paths of integration are indented, if necessary, in such a manner that all poles of $\Gamma(\beta_j - \xi)$, $j = 1, \ldots, m_1$ and $\Gamma(\beta'_k - \eta)$, $k = 1, \ldots, m_2$ lie to the right and those of $\Gamma(\gamma_j + \xi)$, $j = 1, \ldots, \nu_1$, and $\Gamma(\gamma'_k + \eta)$, $k = 1, \ldots, \nu_2$ and $\Gamma(1 - \epsilon_j + \xi + \eta)$, $j = 1, \ldots, n$ lie to the left of the imaginary axis.

The integral (1.1) converges if

$$\left.\begin{array}{l} p + q + s + t < 2(m_1 + \nu_1 + \eta) \\ p + q + s + t < 2(m_2 + \nu_2 + \eta) \\ |\arg x| < \pi\,[m_1 + \nu_1 + n - \tfrac{1}{2}(p + q + s + t)] \\ |\arg y| < \pi\,[m_2 + \nu_2 + n - \tfrac{1}{2}(p + q + s + t)]. \end{array}\right\} \qquad \ldots(1.3)$$

and

2. SIMULTANEOUS TRIPLE INTEGRAL EQUATIONS

The simultaneous triple integral equations to be discussed here are as follows :

$$\int_0^\infty \int_0^\infty G^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)} \left[\begin{array}{c|l} & (\epsilon_{p+n}) \\ x_u & (\gamma^{\theta}_{\nu_1});\ (\gamma'^{\theta}_{\nu_2}) \\ & (\beta_{m_1});\ (\beta'_{m_2}) \\ y_{u'} & (\gamma_{\nu_1+t});\ (\gamma'_{\nu_2+t}) \\ & (\beta^{\theta}_{m_1+q});\ (\beta'^{\theta}_{m_2+q}) \end{array}\right]$$

$$\times \sum_{\psi=1}^{n'} a_{\psi\theta} f_{\psi}(u, u')\, du\, du' = \phi_{1\theta}(x, y), \quad 0 < x < u,\ 0 < y < a \qquad \ldots(2.1)$$

$$\int_0^\infty \int_0^\infty G^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)} \left[\begin{array}{c|l} & (\epsilon_{p+n}) \\ xu & (c_{\nu_1});\ (c'_{\nu_2}) \\ & (\beta_{m_1});\ (\beta'_{m_2}) \\ yu' & (\gamma_{\nu_1+t});\ (\gamma'_{\nu_2+t}) \\ & (b_{m_1+q});\ (b'_{m_2+q}) \end{array}\right]$$

$$\times \sum_{\psi=1}^{n'} b_{\psi\theta} f_{\psi}(u, u')\, du\, du' = \phi_{2\theta}(x, y), \quad a < x < b,\ a < y < b \qquad \ldots(2.2)$$

$$\int_0^\infty \int_0^\infty G^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)} \left[\begin{array}{c|l} & (\epsilon_{p+n}) \\ xu & (c_{\nu_1});\ (c'_{\nu_2}) \\ & (b^{\theta}_{m_1});\ (b'^{\theta}_{m_2}) \\ yu' & (c^{\theta}_{\nu_1+t});\ (c'^{\theta}_{\nu_2+t}) \\ & (b_{m_1+q});\ (b'_{m_2+q}) \end{array}\right] \times$$

(*equation continued on p.* 1602)

$$\times \sum_{\psi=1}^{n'} c_{\psi\theta} f_\psi(u, u')\, du\, du' = \phi_{3\theta}(x, y),$$

$$\theta = 1, 2, \ldots, n, \quad x > b, \quad y > b \qquad \ldots(2.3)$$

where $a_{\psi\theta}$, $b_{\psi\theta}$ and $c_{\psi\theta}$ are constants, $\phi_{1\theta}(x)$, $\phi_{2\theta}(x)$ and $\phi_{3\theta}(x)$ are given and $f_\psi(x, y)$ is to be determined.

3. The Mellin Transform

In this section we use the Mellin type of double integral, if for positive x and y,

$$g_\psi(x, y) = \frac{1}{(2\pi i)^2} \int_{\sigma-i\infty}^{\sigma+i\infty} \int_{\sigma'-i\infty}^{\sigma'+i\infty} x^{-r} y^{-s} f_\psi(r, s)\, dr\, ds \qquad \ldots(3.1)$$

then

$$f_\psi(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} g_\psi(x, y)\, dx\, dy \qquad \ldots(3.2)$$

under suitable conditions due to Reed (1944).

We may write (3.1) and (3.2) in the form :

If

$$M\{f_\psi(x, y)\} = F_\psi(r, s) = \int_0^\infty \int_0^\infty x^{r-1} y^{s-1} g_\psi(x, y)\, dx\, dy, \qquad \ldots(3.3)$$

then

$$M^{-1}\{F_\psi(r, s)\} = g_\psi(x, y) = \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} F_\psi(r, s)\, x^{-r} y^{-s}\, dr\, ds. \qquad \ldots(3.4)$$

We restate here a theorem of the Mellin transform of two variables similar to the Parseval theorem stated by Fox (1965) for single variable:

$$\text{and} \quad \left.\begin{aligned} m\{h(u, u')\} &= H(r, s) \\ M\{f_\psi(xu, yu')\} &= x^{-r} y^{-s} F_\psi(r, s) \end{aligned}\right\} \qquad \ldots(3.5)$$

where

$$m\{f_\psi(u, u')\} = F_\psi(r, s)$$

then

$$\int_0^\infty \int_0^\infty h(xu, yu')\, f_\psi(u, u')\, du\, du'$$

$$= \frac{1}{(2\pi i)^2} \int_{-i\infty}^{i\infty} \int_{-i\infty}^{i\infty} x^{-r} y^{-s} H(r, s)\, F_\psi(1 - r, 1 - s)\, dr\, ds. \qquad \ldots(3.6)$$

From (1.2) and (3.4) it is easily seen that

$$M\left\{ G^{n,\nu_1,\nu_2,m_1,m_2}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)}\left[\begin{array}{c|l} & (\epsilon_{p+n}) \\ x & (\gamma^{\theta}_{\nu_1});\ (\gamma'^{\theta}_{\nu_2}) \\ & (\beta_{m_1});\ (\beta'_{m_2}) \\ y & (\gamma_{\nu_1+t});\ (\gamma'_{\nu_2+t}) \\ & (\beta^{\theta}_{m_1+q});\ (\beta'^{\theta}_{m_2+q}) \end{array}\right]\right\}$$

$$= \frac{\prod_{j=1}^{m_1}\Gamma(\beta_j+r)\prod_{j=1}^{\nu_1}\Gamma(\gamma^{\theta}_j-r)\prod_{j=1}^{m_2}\Gamma(\beta'_j+s)\prod_{j=1}^{\nu_2}\Gamma(\gamma'^{\theta}_j-s)}{\prod_{j=1}^{q}\Gamma(1-\beta^{\theta}_{m_1+j}-r)\prod_{j=1}^{t}\Gamma(1-\gamma_{\nu_1+j}+r)\prod_{j=1}^{q}\Gamma(1-\beta'^{\theta}_{m_2+j}-s)}$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j-r-s)}{\prod_{j=1}^{t}\Gamma(1-\gamma'_{\nu_2+j}+s)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j}+r+s)} \qquad \text{...(3.7)}$$

$$= \chi(r, s). \qquad \text{...(3.8)}$$

On using $M\{f(u, u')\} = F(r, s)$ and applying (3.6) to (2.1), (2.2) and (2.3), we find that

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\chi(r, s)\, x^{-r}, y^{-s}\sum_{\psi=1}^{n'} a_{\psi\theta}F_{\psi}(1-r, 1-s)\, dr\, ds$$

$$= \phi_{1\theta}(x, y),\quad 0 < x < a,\quad 0 < y < a \qquad \text{...(3.9)}$$

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\bar{\chi}(r, s)\, x^{-r}, y^{-s}\sum_{\psi=1}^{n'} b_{\psi\theta}F_{\psi}(1-r, 1-s)\, dr\, ds$$

$$= \phi_{2\theta}(x, y),\quad a < x < b,\quad a < y < b \qquad \text{...(3.10)}$$

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\bar{\bar{\chi}}(r, s)\, x^{-r}, y^{-s}\sum_{\psi=1}^{n'} c_{\psi\theta}F_{\psi}(1-r, 1-s)\, dr\, ds$$

$$= \phi_{3\theta}(x, y),\quad x > b,\quad y > b \qquad \text{...(3.11)}$$

where $\theta = 1, 2, 3, \ldots, n,$

$$\bar{\chi}(r, s) = \frac{\prod_{j=1}^{m_1} \Gamma(\beta_j + r) \prod_{j=1}^{\nu_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(\beta'_j + s) \prod_{j=1}^{\nu_2} \Gamma(c'_j - s)}{\prod_{j=1}^{q} \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^{t} \Gamma(1 - \gamma_{\nu_1+j} + r) \prod_{j=1}^{q} \Gamma(1 - b'_{m_2+j} - s)}$$

$$\times \frac{\prod_{j=1}^{n} \Gamma(1 - \epsilon_j - r - s)}{\prod_{j=1}^{t} \Gamma(1 - \gamma'_{\nu_2+j} + s) \prod_{j=1}^{p} \Gamma(\epsilon_{n+j} + r + s)} \qquad \ldots(3.12)$$

$$\bar{\bar{\chi}}(r, s) = \frac{\prod_{j=1}^{m_1} \Gamma(b^{g}_j + r) \prod_{j=1}^{\nu_1} \Gamma(c_j - r) \prod_{j=1}^{m_2} \Gamma(b'^{\theta}_j + s) \prod_{j=1}^{\nu_2} \Gamma(c'_j - s)}{\prod_{j=1}^{q} \Gamma(1 - b_{m_1+j} - r) \prod_{j=1}^{t} \Gamma(1 - c\gamma^{\theta}_{\nu_1+j} + r) \prod_{j=1}^{q} \Gamma(1 - b'_{m_2+j} - s)}$$

$$\times \frac{\prod_{j=1}^{n} \Gamma(1 - \epsilon_j - r - s)}{\prod_{j=1}^{t} \Gamma(1 - c'^{\theta}_{2+j} + s) \prod_{j=1}^{p} \Gamma(\epsilon_{n+j} + r + s)} \qquad \ldots(3.13)$$

and $\chi(r, s)$ is given by (3.8).

4. SOLUTION OF SIMULTANEOUS TRIPLE INTEGRAL EQUATIONS

In this section we will transform (3.9) and (3.11) into two others with common kernel by application of fractional integration operators of two variables similar for one variable defined by Erdelyi (1950–51).

$$T_{1,1}[\alpha, \beta, \alpha', \beta' : r, s : w(x, y)] = \{rs/\Gamma(\alpha)\,\Gamma(\alpha')\}\, x^{-r\alpha+r-\beta-1}$$

$$\times y^{-s\alpha'+s-\beta'-1} \int_0^x \int_0^y (x^r - v^r)^{\alpha-1} (y^s - v'^s)^{\alpha'-1} v^{\beta} v'^{\beta'} w(v, u')\, dv\, dv' \qquad \ldots(4.1)$$

$$R_{1,1}[\alpha, \beta, \alpha', \beta' : r, s : w(x, y)] = \{rs/\Gamma(\alpha)\,\Gamma(\alpha')\}\, x^{\beta} y'^{\beta'}$$

$$\times \int_x^\infty \int_y^\infty (v^r - x^r)^{\alpha-1} (v'^s - y^s)^{\alpha'-1} v^{-\beta-r\alpha+r-1} v'^{-\beta'-s\alpha'+s-1}$$

$$\times w(v, v')\, dv\, dv'. \qquad \ldots(4.2)$$

In the contracted forms we write

$$T[(\gamma_j - c_j)(\gamma'_j - c'_j);\, c_{j-1}, c'_j - 1 : 1, 1 : w(x, y)] = T_j[w(x, y)] \qquad \ldots(4.3)$$

$$T[(\beta_{m_1+k} - b_{m_1+k}), (\beta'_{m_2+k} - b'_{m_2+k});\, -B'_{m_1+k} - \beta'_{m_2+k} : 1, 1 : w(x, y)]$$

$$= T^{*1}_k[w(x, y)] \qquad \ldots(4.4)$$

$$R[b_l - \beta_l), (b'_l - \beta'_l);\, \beta_l, \beta'_l : 1, 1 : w(x, y)] = R_l[w(x, y)] \qquad \ldots(4.5)$$

$$R\,[(c_{\nu_1+h} - \gamma_{\nu_1+h}),\ (c'_{\nu_2+h} - \gamma'_{\nu_2+h});\ c'_{\nu_1+h},\ c'_{\nu_2+h} : 1,\ 1 : w(x, y)]$$

$$= R_h^{*2}[w(x, y)]. \qquad \ldots(4.6)$$

Now in (3.9), replace x and y and v and v' respectively, multiply by

$$(x - v)^{\gamma_{\nu_1} - c_{\nu_1 - 1}}\,(y - v')^{\gamma'_{\nu_2} - c'_{\nu_2 - 1}}\,v^{c_{\nu_1 - 1}}\,v'^{c'_{\nu_2 - 1}}$$

and integrable through the integral sign w, r, t, v and v' from 0 to x and 0 to y respectively, where $0 < x < a$, $0 < y < a$. We obtain

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j + r)\prod_{j=1}^{\nu_1 - 1}\Gamma(\gamma_j^{\theta} - r)\prod_{j=1}^{m_2}\Gamma(\beta'_j + s)\prod_{j=1}^{\nu_2 - 1}\Gamma(\gamma_j'^{\theta} - s)}{\prod_{j=1}^{q}\Gamma(1 - \beta^{\theta}_{m_1+j} - r)\prod_{j=1}^{t}\Gamma(1 - \gamma_{\nu_1+j} + r)\prod_{j=1}^{q}\Gamma(1 - \beta'^{\theta}_{m_2+j} - s)}$$

$$\times\frac{\Gamma(c_{\nu_1} - r)\,\Gamma(c'_{\nu_2} - s)\prod_{j=1}^{n}\Gamma(1 - \epsilon_j - r - s)}{\prod_{j=1}^{t}\Gamma(1 - \gamma'_{\nu_2+j} + s)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j} + r + s)}\,x^{-r}y^{-s}$$

$$\times\sum_{\psi=1}^{n'} a_{\psi\theta}F_{\psi}(1 - r,\ 1 - s)\,dr\,ds$$

$$= \frac{x^{1-\gamma^{\theta}_{\nu_1}}\,y^{1-\gamma'^{\theta}_{\nu_2}}}{\Gamma(\gamma^{\theta}_{\nu_1} - c_{\nu_1})\,\Gamma(\gamma'^{\theta}_{\nu_2} - c'_{\nu_2})}\int_0^x\int_0^y (x - v)^{\gamma^{\theta}_{\nu_1} - c_{\nu_1} - 1}\,(y - v')^{\gamma'^{\theta}_{\nu_2} - c_{\nu_2} - 1}$$

$$\times\, v^{c_{\nu_1} - 1}\,v'^{c'_{\nu_2} - 1}\,\phi_{1\theta}(v, v')\,dv\,dv' = T_{\nu_1, \nu_2}\,[\phi_{1\theta}(v, v')] \qquad \ldots(4.7)$$

where $0 < x < a$, $0 < y < a$, by virtue of (4.1) and (4.3).

On transforming (4.7) successively, for ($k = \nu_1 - 1, \ldots, 1$; $\nu_2 - 1, \ldots, 1$) by the application of operators $T_{k,k}$ and $T^{*1}_{k,k}$ ($k = q, q - 1, \ldots, 1$), we finally get

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j + r)\prod_{j=1}^{\nu_1 - 1}\Gamma(c_j - r)\prod_{j=1}^{m_2}\Gamma(\beta'_j + s)\prod_{j=1}^{\nu_2}\Gamma(c'_j - s)}{\prod_{j=1}^{q}\Gamma(1 - b_{m_1+j} - r)\prod_{j=1}^{t}\Gamma(1 - \gamma_{\nu_1+j} + r)\prod_{j=1}^{q}\Gamma(1 - b'_{m_2+j} - s)}$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1 - \epsilon_j - r - s)}{\prod_{j=1}^{t}\Gamma(1 - \gamma'_{\nu_2+j} + s)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j} + r + s)}$$

(*equation continued on p. 1606*)

$$\times\ x^{-r}y^{-s}\sum_{\psi=1}^{n'} b_{\psi\theta}F_{\psi}(1-r,\ 1-s)\,dr\,ds$$

$$= \sum_{\psi=1}^{n'} d_{\psi\theta}T^{*1}_{1,1}\,[T^{*1}_{2,2}\ \ldots\ T^{*1}_{q,q}\,T_{1,1}\ \ldots\ T_{\nu_1,\nu_2}\,\{\phi_{1\theta}(x,y)\}\ \ldots]$$

$$0 < x < a,\quad 0 < y < a \qquad \ldots(4.8)$$

where $d_{\psi\theta}$ are the elements of the matrix $[b_{\psi\theta}]\,[a_{\psi\theta}]^{-1}$, $\theta = 1, 2, \ldots, n$.

In a similar manner by the application of operators R_l and R^{*2}_h given in (4.5) and (4.6) respectively for $l = m_1, m_1 - 1, \ldots, 1;\ m_2, m_2 - 1, \ldots, 1$ and $h = t, t - 1, \ldots, 1$ to (3.11) it can be easily seen that it transforms into the desired form

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j+r)\prod_{j=1}^{\nu_1}\Gamma(c_j-r)\prod_{j=1}^{m_2}\Gamma(\beta_j'+s)\prod_{j=1}^{\nu_2}\Gamma(c_j'-s)}{\prod_{j=1}^{q}\Gamma(1-b_{m_1+j}-r)\prod_{j=1}^{t}\Gamma(1-\gamma_{\nu_1+j}+r)\prod_{j=1}^{q}\Gamma(1-b'_{m_2+j}-s)}$$

$$\times\ \frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j-r-s)}{\prod_{j=1}^{t}\Gamma(1-\gamma'_{\nu_2+j}+s)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j}+r+s)}$$

$$\times\ x^{-r}y^{-s}\prod_{\psi=1}^{n'} b_{\psi\theta}F_{\psi}(1-r,\ 1-s)\,dr\,ds$$

$$= \sum_{\psi=1}^{n'} g_{\psi\theta}R^{*2}_{1,1}\,[R^{*2}_{2,2}\ \ldots\ R^{*2}_{t,t}\,R_{1,1}\ \ldots\ R_{m_1,m_2}\,\{\phi_{3\theta}(x,y)\}\ \ldots],$$

$$x > b,\quad y > b \qquad \ldots(4.9)$$

where $g_{\psi\theta}$ are the elements of the matrix $[b_{\psi\theta}]\,[c_{\psi\theta}]^{-1}$, $\theta = 1, 2, \ldots, n$.

On setting

$$G_\theta(x,y) = \begin{cases} = \sum_{\psi=1}^{n'} d_{\psi\theta}T^{*1}_{1,1}\,[T^{*}_{2,2}\ \ldots\ T^{*1}_{q,q}\,T_{1,1}\ \ldots\ T_{\nu_1,\nu_2}\,\{\phi_{1\theta}(x,y)\ \ldots], \\ \qquad\qquad 0 < x < a,\quad 0 < y < a \\ \{\phi_{2\theta}(x,y)\},\ a < x < b,\ a < y < b, \\ \sum_{\psi=1}^{n'} g_{\psi\theta}R^{*2}_{1,1}\,[R^{*2}_{2,2}\ \ldots\ R^{*2}_{t,t}\,R_{1,1}\ \ldots\ R_{m_1,m_2}\,\{\phi_{3\theta}(x,y)\}\ \ldots], \\ \qquad\qquad x > b,\quad y > b \end{cases} \qquad \ldots(4.10)$$

where $d_{\psi\theta}$ and $g_{\psi\theta}$ are the elements of the matrix $[b_{\psi\theta}]\,[a_{\psi\theta}]^{-1}$, $[b_{\psi\theta}]\,[c_{\psi\theta}]^{-1}$ respectively.

Since the left-hand side of (4.8), (4.9) and (3.10) are same with a common kernel they can be put into a compact form as

$$\frac{1}{(2\pi i)^2}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{m_1}\Gamma(\beta_j+r)\prod_{j=1}^{\nu_1}\Gamma(c_j-r)\prod_{j=1}^{m_2}\Gamma(\beta'_j+s)\prod_{j=1}^{\nu_2}\Gamma(c'_j-s)}{\prod_{j=1}^{q}\Gamma(1-b_{m_1+j}-r)\prod_{j=1}^{t}\Gamma(1-\gamma_{\nu_1+j}+r)\prod_{j=1}^{q}\Gamma(1-b'_{m_2+j}-s)}$$

$$\times\frac{\prod_{j=1}^{n}\Gamma(1-\epsilon_j-r-s)}{\prod_{j=1}^{t}\Gamma(1-\gamma'_{\nu_2+j}+s)\prod_{j=1}^{p}\Gamma(\epsilon_{n+j}+r+s)}\,x^{-r}y^{-s}$$

$$\times\sum_{\psi=1}^{n'} b_{\psi\theta}F_{\psi}(1-r,1-s)\,dr\,ds=G_{\theta}(x,y). \qquad \ldots(4.11)$$

Equation (4.11) is the reduction of (3.9) and (3.11) to two equations with a common kernel. On treating the kernel of (4.11) as an unsymmetrical Fourier kernel and following a procedure similar to that of one variable by Fox (1965), (4.11) can be written as

$$f_{\psi}(x,y)=\frac{1}{(2\pi i)^2}\sum_{\psi=1}^{n'}\Theta_{\psi\theta}\int_{-i\infty}^{i\infty}\int_{-i\infty}^{i\infty}\frac{\prod_{j=1}^{q}\Gamma(1-b_{m_1+j}+r)\prod_{j=1}^{t}\Gamma(2-\gamma_{\nu_1+j}-r)}{\prod_{j=1}^{m_1}\Gamma(1+\beta_j-r)\prod_{j=1}^{\nu_1}\Gamma(-1+c_j+r)}$$

$$\times\frac{\prod_{j=1}^{q}\Gamma(-b'_{m_2+j}+s)\prod_{j=1}^{t}\Gamma(2-\gamma'_{\nu_2+j}-s)\prod_{j=1}^{p}\Gamma(2+\epsilon_{n+j}-r-s)}{\prod_{j=1}^{m_2}\Gamma(1-\beta'_j-s)\prod_{j=1}^{\nu_2}\Gamma(-1+c'_j+s)\prod_{j=1}^{n}\Gamma(-1-\epsilon_j+r+s)}$$

$$\times\, x^{-r}y^{-s}G'_{\theta}(1-r,1-s)\,dr\,ds \qquad \ldots(4.12)$$

where $m\,\{G_{\theta}(x,y)\}=G'_{\theta}(r,s)$.

This is the formal solution of (2.1), (2.2) and (2.3). Applying (3.6) to (4.12), we see that

$$f_{\psi}(x,y)=\sum_{\psi=1}^{n'}\Theta_{\psi\theta}\int_0^{\infty}\int_0^{\infty}G^{p,t,t,q,q}_{p+n,(t+\nu_1 t+\nu_2),0,(q+m_1,q+m_2)}$$

$$\times\left[\begin{matrix}xu\\ yu'\end{matrix}\left|\begin{matrix}(-1-\epsilon_{n+1},p);(-1-\epsilon_n)\\ (2-\gamma_{\nu_1}+1,t);(2,c_{\nu_1});(2-\gamma'_{\nu_2+1},t)\,(2-c'_{\nu_2})\\ (-b_{m_1+1},q);(-\beta_{m_1});(-b'_{m_2+1},q),(-\beta'_{m_2})\end{matrix}\right.\right]$$

$$\times\, G_{\theta}(u,u')\,du\,du' \qquad \ldots(4.13)$$

where $\Theta_{\psi\theta}$ are the elements of the matrix $[b_{\psi\theta}]^{-1}$, $\theta = 1, 2, \ldots, n$. When (4.13) written out in full becomes :

$$\begin{aligned}
f_\psi(x, y) = \sum_{\psi=1}^{n'} \Theta_{\psi\theta} \Bigg[& \int_0^a \int_0^a G^{p,t,t,q,q}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)} \begin{bmatrix} xu \\ yu' \end{bmatrix} \\
& \times \sum_{\psi=1}^{n} d_{\psi\theta} T^{*1}_{1,1} [T^{*1}_{2,2}, \ldots, T^{*1}_{q,q}\, T_{1,1}, \ldots, T_{\nu_1,\nu_2} \{\phi_{1\theta}(u, u')\} \ldots]\, du\, du' \\
& + \int_a^b \int_a^b G^{p,t,t,q,q}_{p+n,(t+\nu_1,t+\nu_2),0,(q+m_1,q+m_2)} \begin{bmatrix} xu \\ yu' \end{bmatrix} \{\phi_{2\theta}(u, u')\} \ldots du\, du' \\
& + \int_b^\infty \int_b^\infty G^{p,t,t,q,q}_{p+n\cdot(t+\nu_1,t+\nu_2),0(q+m_1,q+m_2)} \begin{bmatrix} xu \\ yu' \end{bmatrix} \\
& \times \sum_{\psi=1}^{n'} g_{\psi\theta} R^{*2}_{1,1} [R^{*2}_{2,2} \ldots R^{*2}_{t,t}\, R_{1,1} \ldots R_{m_1,m_2} \{\phi_{3\theta}(u, u')\} \ldots]\, du\, du' \Bigg].
\end{aligned} \quad \ldots(4.14)$$

5. A PARTICULAR CASE

If we take $a = b = 1$ and $n' = \theta = 1$, then the simultaneous equations (2.1) to (2.3) reduce to those of dual equations discussed earlier by Masood and Srivastava (1970) and our solution (4.14) agrees with that of Masood and Srivastava (1970).

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