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THE PSEUDO-DIFFERENTIAL OPERATOR $h_{\mu, a}$ ON SOME GEVREY SPACES

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Continuity of the pseudo-differential operator $h_{\mu, a}$ is investigated on certian Gevrey spaces when the symbol of the pseudo-differential operator also belongs to a certain Gevrey class.

Key Words : Pseudo-Differential Operator; Hankel Transform; Bessel-Differential Operator; Gevrey Space

1. INTRODUCTION

The Hankel transformation

$$\Phi(y) = (h_\mu \phi)(y) = \int_0^\infty \sqrt{xy}\, J_\mu(xy)\, \phi(x)\, dx, \quad \mu \geq -1/2. \qquad \dots (1.1)$$

is known to be an automorphism on the Zemanian space $H_\mu(I)$, $I = (0, \infty)$, consisting of all complex-valued infinitely differentiable functions ϕ on I which satisfy :

$$\gamma^\mu_{m,k}(\phi) = \sup_{x \in I} \left| x^m (x^{-1} D)^k x^{-\mu - 1/2} \phi(x) \right| < \infty \qquad \forall m, k \in \mathbf{N}_0 \qquad \dots (1.2)$$

Using theory of H_μ space [7] Pathak and Pandey [4] investigated the pseudo-differential operator $h_{\mu, a}$ defined as follows :

$$(h_{\mu, a} \phi)(x) = \int_0^\infty \sqrt{xy}\, J_\mu(xy)\, a(x, y)\, (h_\mu \phi)(y) dy \quad \mu \geq -1/2. \qquad \dots (1.3)$$

With symbol $a(x, y)$ satisfying certain growth conditions $h_{\mu, a}$ is a continuous linear map of $H_\mu(I)$ into itself [4].

In this paper we assume that the symbol $a(x, y)$ belongs to a certain Gevrey class $S^{\infty, \omega}_{\rho, \delta}$ and study the pseudo-differential operator $h_{\mu, a}$ on Gevrey function spaces introduced by Pathak and Shrestha [3].

2. THE SPACE $H_\mu(\omega)$

Let ω be a continuous real-valued function defined on $I = (0, \infty)$ such that $\omega(0) = 0$ and

$$(a) \;\; 0 \leq \omega(x + y) \leq \omega(x) + \omega(y) \qquad \forall x, y \in I \qquad \dots (2.1)$$

$$(b) \int_0^\infty \frac{\omega(x)dx}{1+x^2} < \infty \qquad \text{... (2.2)}$$

$$(c)\ \omega(x) \geq a + b \log(1+x) \text{ for some real } a \text{ and } b > 0. \qquad \text{... (2.3)}$$

The class of all such ω functions is denoted by M.

Now assume that ω is a function in M. A function ϕ is said to be in the space $H_\mu(\omega)$, where ϕ and $h_\mu(\phi)$ are smooth functions and for every $\mu \in \mathbf{R}, n \in \mathbf{N}_0$ and m a positive real number,

$$\alpha^\mu_{m,n}(\phi) = \sup_{x \in (0,\infty)} e^{m\omega(x)} \left| (x^{-1}D)^n x^{-\mu-1/2} \phi(x) \right| < \infty \qquad \text{... (2.4)}$$

and

$$\beta^\mu_{m,n}(\phi) = \sup_{x \in (0,\infty)} e^{m\omega(x)} \left| (x^{-1}D)^n x^{-\mu-1/2} (h_\mu\phi)(x) \right| < \infty. \qquad \text{... (2.5)}$$

On $H_\mu(\omega)$ we consider the topology generated by the family

$$\left\{ \alpha^\mu_{m,n}, \beta^\mu_{m,n} \right\}_{m \in R_+, n \in N_0}$$

of seminorms. Various properties of the space $H_\mu(\omega)$ can be found in [3]. In what follows we study the pseudo-differential operator $h_{\mu,a}$ of infinite order on $H_\mu(\omega)$. For this purpose we define the symbol class $S^{\infty,\omega}_{\rho,\delta}$.

Definition 2.1 — The function $a(x,y) : C^\infty(I \times I) \to \mathbf{C}$ belongs to class $S^{\infty,\omega}_{\rho,\delta}$ if and only if $\forall p, q \in \mathbf{N}_0, m \in \mathbf{R}_+$, and $\sigma > 0$,

$$e^{m\omega(x)} \left| (x^{-1}\partial/\partial x)^p (y^{-1}\partial/\partial y)^q y^{2\mu+1} a(x,y) \right|$$
$$\leq c(\sigma, m)\, C^{p+q} p!q!\, (1+y)^{-\rho q + \delta p} e^{\sigma\omega(y)}, \qquad \text{... (2.6)}$$

where C is a constant, ρ, δ are real numbers such that $0 \leq \delta < \rho < 1$.

The previous Definition 2.1 is suggested by similar classes of the symbols in the literature, See[5]. Here we shall limit ourselves to give two theorems of continuity on $H_\mu(\omega)$ and related spaces $G_\mu(\omega)$, which are valid under assumptions somewhat weaker than (2.6), cf. the next proofs.

***Theorem* 2.2** — *The pseudo-differential operator $h_{\mu,a}$ is a continuous linear map of $H_\mu(\omega)$ into itself for $\mu \geq -1/2$.*

PROOF : (*i*) Let $\phi \in H_\mu(\omega)$. Then in view of Definition (1.3) using Leibnitz type formula [2, p. 242]

$$(x^{-1}D)^k (x^{-\mu-1/2}\psi\phi) = \sum_{\nu=0}^{k} \binom{k}{\nu} (x^{-1}D)^\nu \psi (x^{-1}D)^{k-\nu} x^{-\mu-1/2}\phi \qquad \text{... (2.7)}$$

we have

$$e^{m\omega(x)} \left| (x^{-1} D)^n x^{-\mu - 1/2} (h_{\mu, a} \phi)(x) \right| = e^{m\omega(x)}$$

$$\times \left| (x^{-1} D)^n x^{-\mu - 1/2} \int_0^\infty (xy)^{1/2} J_\mu (xy) a(x, y) (h_\mu \phi)(y) dy \right|$$

$$= e^{m\omega(x)} \left| (x^{-1} D)^n \int_0^\infty (xy)^{-\mu} J_\mu (xy) y^{2\mu + 1} a(x, y) y^{-\mu - 1/2} (h_\mu \phi)(y) dy \right|$$

$$= \int_0^\infty \sum_{r=0}^n \binom{n}{r}$$

$$\times \left| (x^{-1} D)^{n-r} [(xy)^{-\mu} J_\mu (xy)] e^{m\omega(x)} (x^{-1} D)^r [y^{2\mu + 1} a(x, y)] y^{-\mu - 1/2} (h_\mu \phi)(y) \right| dy.$$

Using inequality (2.6), the last expression can be estimated by

$$\int_0^\infty \sum_{r=0}^n \binom{n}{r} y^{2(n-r)} \left| (xy)^{-\mu - n + r} J_{\mu + n - r} (xy) \right|$$

$$\times c(\sigma, m) C^r r! (1 + y)^{r\delta} e^{\sigma\omega(y)}$$

$$\times e^{-m\omega(y)} \left| e^{m\omega(y)} y^{-\mu - 1/2} (h_\mu \phi)(y) \right| dy. \qquad \text{... (2.8)}$$

By property (c) of the function $\omega(x)$ we have

$$(1 + y) \le e^{-a/b} e^{\omega(y)/b}, \qquad \text{... (2.9)}$$

and from [2, p. 247] we know that there exists a constant Q_μ independent of r such that

$$\left| z^{-\mu - n + r} J_{\mu + n - r}(z) \right| \le Q_\mu.$$

Therefore (2.8) can be bounded by

$$Q_\mu c(\sigma, m) \sum_{r=0}^n \binom{n}{r} C^r r!$$

$$\times \int_0^\infty e^{(\sigma - m)\omega(y)} e^{-ar\delta/b} e^{r\delta\omega(y)/b} y^{2(n-r)} e^{m\omega(y)}$$

$$\times \left| y^{-\mu - 1/2} (h_\mu \phi)(y) \right| dy$$

$$\le Q_\mu c(\sigma, m) \sum_{r=0}^n \binom{n}{r} C^r r!$$

$$\int_0^\infty e^{(\sigma - m)\omega(y)} y^{2(n-r)} e^{-ar\delta/b} \sup_{y \in I} e^{(m + r\delta/b)\,\omega(y)}$$

$$\left| y^{-\mu - 1/2} (h_\mu \phi)(y) \right| dy.$$

Using (2.5), we have

$$\sup_{x \in I} e^{m\omega(x)} \left| (x^{-1} D)^n x^{P-\mu - 1/2} (h_{\mu, a} \phi)(x) \right|$$

$$\leq Q_\mu c(\sigma, m) \sum_{r=0}^{n} \binom{n}{r} C^r r!\, e^{-ar\delta/b}$$

$$\times \beta^{\mu}_{(m + r\delta/b), 0} (\phi) \int_0^\infty e^{(\sigma - m)\omega(y)} y^{2(n-r)} dy < \infty$$

by choosing $m > \sigma$. Therefore, $\alpha^{\mu}_{m, n} (h, \mu, a\ \phi) < \infty$.

(*ii*) Let $\Phi(x) = (h_{\mu, a} \phi)(x)$. Then to complete the proof of the theorem we need to show that $\Phi(x)$ satisfies (2.5). From property (*b*) of the function $\omega(x)$ it follows that to every $\varepsilon > 0$ there exists a constant $c(\varepsilon)$ such that

$$\omega(x) \leq \varepsilon x + c(\varepsilon); \qquad \text{... (2.10)}$$

so that

$$e^{m\omega(x)} \leq e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} x^\nu. \qquad \text{... (2.11)}$$

Therefore, following the technique of Zemanian [7, p. 141] we can write

$$e^{m\omega(x)} \left| (x^{-1} D)^n x^{-\mu - 1/2} (h_\mu \Phi)(x) \right|$$

$$\leq e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} x^\nu \left| (x^{-1} D)^n x^{-\mu - 1/2} (h_\mu \Phi)(x) \right|$$

$$\leq e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!}$$

$$\left| \int_0^\infty y^{2\mu + 2n + \nu + 1} (y^{-1} D)^\nu [y^{-\mu - 1/2} \Phi(y)]\, (xy)^{-(\mu + n)} J_{\mu + \nu + n}(xy)\, dy \right|. \qquad \text{... (2.12)}$$

Since for $\mu \geq -1/2$, $\left| (xy)^{-(\mu + n)} J_{\mu + \nu + n}(xy) \right|$ is bounded on $0 < x, y < \infty$ by Q_μ, applying definition (1.3), right-hand side of the eq. (2.12) can be estimated by

$$e^{mc(\varepsilon)} Q_\mu \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty y^{2\mu+2n+\nu+1}$$

$$\times \left| (y^{-1} D)^\nu \left[y^{-\mu-1/2} \int_0^\infty (yt)^{1/2} J_\mu (yt) a(y,t)(h_\mu \phi)(t) dt \right] \right| dy$$

$$\leq e^{mc(\varepsilon)} Q_\mu \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty y^{2\mu+2n+\nu+1}$$

$$\times \sum_{s=0}^{\nu} \binom{\nu}{s} \left| (y^{-1} D)^{\nu-s} \left[(yt)^{-\mu} J_\mu (yt) \right] \right|$$

$$\times \left| (y^{-1} D)^s \left[t^{2\mu+1} a(y,t) \right] t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt\, dy$$

$$\leq e^{mc(\varepsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty y^{2\mu+2n+\nu+1}$$

$$\times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y,t) \right| \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt\, dy. \quad \text{... (2.13)}$$

Suppose N is an integer no less than $2\mu + 2n + 1$, then

$$y^{2\mu+2n+\nu+1} \leq (1+y)^{N+\nu}. \quad \text{... (2.14)}$$

Using (2.14), right-hand side of eq. (2.13) can be bounded by

$$Q_\mu^2 e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty (1+y)^{N+\nu+2}$$

$$\times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y,t) \right|$$

$$\times \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \frac{dy}{(1+y)^2}.$$

Using the inequality (2.3) and $(1 + y) \leq e^{-a/b} e^{\omega(y)/b}$, the right-hand side can be bounded by

$$Q_\mu^2 e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty \exp(-a(N+\nu+2)/b) \exp[(N+\nu+2)\omega(y)/b]$$

$$\times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right|$$

$$\times \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \frac{dy}{(1+y)^2}$$

$$\leq Q_\mu^2 e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \exp[-a(N+\nu+2)/b]$$

$$\sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \sup_{y \in I} \exp[(N+\nu+2)\omega(y)/b]$$

$$\times \left| (y^{-1} D)^s t^{2\mu+1} a(y, t) \right| \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \int_0^\infty \frac{dy}{(1+y)^2}. \quad \dots (2.15)$$

Using inequality (2.6) the right-hand side can be bounded by

$$Q_\mu^2 e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} \int_0^\infty \int_0^\infty \exp(-a(N+\nu+2)/b)$$

$$\times \sum_{s=0}^{\nu} \binom{\nu}{s} t^{2(\nu-s)} \int_0^\infty c(\sigma, (N+\nu+2)/b) C^s s! (1+t)^{s\delta}$$

$$\times e^{\sigma\omega(t)} \left| t^{\mu-1/2} (h_\mu \phi)(t) \right| dt.$$

Since $s\delta \leq \nu$ the above expression can be estimated by

$$Q_\mu^2 e^{mc(\varepsilon)} \exp(-a(N+2)/b)$$

$$\times \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} e^{-a\nu/b} c(\sigma, (N+\nu+2)/b) \sum_{s=0}^{\nu} C^s s!$$

$$\times \sup_{t \in I} \left| (1+t)^{3\nu+2} e^{\sigma\omega(t)} \left[t^{-\mu-1/2} (h_\mu \phi)(t) \right] \right| \int_0^\infty \frac{dt}{(1+t)^2}$$

$$\leq Q_\mu^2 \exp(mc(\varepsilon) - a(N+2)/b) \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} e^{-a\nu/b}$$

$$\times c(\sigma, (N+\nu+2)/b) \sum_{s=0}^{\nu} \binom{\nu}{s} C^s s!$$

$$\times e^{-a(3\nu+2)/b} \sup_{t\in I} \left| \exp\left(((3\nu+2)/b+\sigma)\,\omega(t)\right)\left[t^{-\mu-1/2} (h_\mu \phi)(t) \right]\right| . \qquad \text{... (2.16)}$$

Using property (2.5) we can estimate the right-hand side of (2.16) by

$$Q_\mu^2 \exp\left(mc(\varepsilon) - a(N+4)/b\right) \sum_{\nu=0}^{\infty} \frac{(m\varepsilon)^\nu}{\nu!} e^{-4a\nu/b}$$

$$\times\ c(\sigma, (N+\nu+2)/b) \sum_{s=0}^{\nu} \binom{\nu}{s} C^s\, \nu!$$

$$\times\ \beta^{\mu}_{\sigma+(3\nu+2)/b,\,0} (h_\mu \phi)$$

$$\leq Q_\mu^2 \exp\left(mc(\varepsilon) - a(N+4)/b\right) \sum_{\nu=0}^{\infty} (m\varepsilon)^\nu$$

$$\times\ c(\sigma, (N+\nu+2)/b)\, e^{-4a\nu/b} \sum_{s=0}^{\nu} \binom{\nu}{s} C^s$$

$$\times\ \beta^{\mu}_{\sigma+(3\nu+2)/b,\,0} (h_\mu \phi)$$

$$\leq Q_\mu^2 \exp\left(mc(\varepsilon) - a(N+4)/b\right) \sum_{\nu=0}^{\infty} [[(m\varepsilon)\, e^{-4a/b} (1+C)]^\nu$$

$$\times \left\{ \beta^{\mu}_{\sigma+(3\nu+2)/b,\,0} (h_\mu \phi)\, c(\sigma, (N+\nu+2)/b) \right\}^{1/\nu}]^\nu.$$

Now choosing

$$\varepsilon < \left\{ \beta^{\mu}_{\sigma+(3\nu+2)/b,\,0} (h_\mu \phi)\, c(\sigma, N+\nu+2)/b \right\}^{-1/\nu} (m e^{-4a/b} (1+C))^{-1},$$

we find that the last series is convergent. This conclues the proof.

3. The Space $G_\mu(\omega)$

The Bessel-differential operator S_μ is defined by

$$S_{\mu,x} = d^2/dx^2 + (1-4\mu)^2/4x^2. \qquad \text{... (3.1)}$$

From [7, p. 139] we know that for any $\phi \in H_\mu(I)$,

$$h_\mu (S_\mu \phi) = -y^2\, h_\mu \phi \qquad \text{... (3.2)}$$

and

$$S^n_{\mu,x}\,\phi(x) = \sum_{j=0}^{n} b_j x^{2j+\mu+1/2} (x^{-1}D)^{n+j} (x^{-\mu-1/2}\phi(x)) \qquad \text{... (3.3)}$$

where b_j are constants depending only on μ.

Now assume that ω is a function in M. A function $\phi \in C^{\infty}(I)$ is said to be in the space $G_\mu(\omega)$ if for every $\mu \in \mathbf{R}, n \in \mathbf{N}_0$, and m a positive real number,

$$A^{\mu}_{m,n}(\phi) = \sup_{x \in (0,\infty)} e^{m\omega(x)} \left| S^n_\mu \phi(x) \right| < \infty \qquad \text{... (3.4)}$$

and

$$A^{\mu}_{m,n}(\phi) = \sup_{x \in (0,\infty)} e^{m\omega(x)} \left| S^n_\mu (h_\mu \phi)(x) \right| < \infty. \qquad \text{... (3.5)}$$

The family $\left\{ A^{\mu}_{m,n}, B^{\mu}_{m,n} \right\}_{m \in R,\, n \in N_0}$ of seminorms generates the topology of $G_\mu(\omega)$ can be in[1]

***Theorem* 3.1** — *The pseudo-differential operator $h_{\mu,a}$ is a continuous linear map of $G_\mu(\omega)$ into itself for $\mu \ge -1/2$.*

PROOF (*i*) : Let $\phi \in G_\mu(\omega)$ then in view of formula (3.3) we have

$$e^{m\omega(x)} \left| S^n_\mu (h_{\mu,a}\phi)(x) \right| = e^{m\omega(x)}$$

$$\times \left| \sum_{j=0}^{n} b_j x^{2j+\mu+1/2} (x^{-1}D)^{n+j} x^{-\mu-1/2} (h_{\mu,a}\phi)(x) \right|.$$

Using definiton (1.3) we can show that the right-hand side is bounded by

$$\sum_{j=0}^{n} |b_j| x^{2j+\mu+1/2} \int_0^{\infty} \sum_{r=0}^{n+j} \binom{n+j}{r}$$

$$\times \left| (x^{-1}D)^{n+j-r} \left[(xy)^{-\mu} J_\mu(xy) \right] \right| e^{m\omega(x)} \left| (x^{-1}D)^r y^{2\mu+1} a(x,y) \right|$$

$$\times \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy.$$

Since by property (c), $x \le (1+x) \le e^{-a/b} e^{\omega(x)/b}$, using inequality (2.6) the above expression can be estimated by

$$c(\sigma, m)\, e^{-(\mu+1/2)a/b} Q_\mu \int_0^{\infty} \sum_{j=0}^{n} \sum_{r=0}^{n+j} |b_j|$$

$$\times \binom{n+j}{r} e^{-2aj/b} y^{2(n+j-r)} C^r r!\, e^{-r\delta a/b}$$

$$\times e^{2n\delta\omega(y)/b}\, e^{(\sigma-m)\omega(y)}\, e^{m\omega(y)} \left| y^{-\mu-1/2} (h_\mu \phi)(y) \right| dy$$

$$\le c(\sigma, m)\, e^{-(\mu+1/2)/b}\, Q_\mu \int_0^\infty \sum_{j=0}^{n} \sum_{r=0}^{n+j} |b_j|$$

$$\times \binom{n+j}{r} e^{-2aj/b}\, e^{-r\delta a/b}\, C^r\, r!\, \beta^{\mu}_{m+2n\delta/b,\, 0}(\phi)$$

$$\times \int_0^\infty e^{(\sigma-m)\omega(y)}\, y^{2(n+j-r)}\, dy < \infty, \qquad \text{... (3.6)}$$

choosing $m > \sigma$. Therefore,

$$A^{\mu}_{m,n}(h_{\mu,a}\,\phi) < \infty.$$

(*ii*) Let $\Phi(x) = (h_{\mu,a}\,\phi)(x)$, then as in the proof of Theorem 2.2 (*ii*), using inequality (2.11), we have

$$e^{m\omega(x)} \left| S^n_\mu (h_\mu \Phi)(x) \right| \le e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} |b_j|$$

$$\times \frac{(m\varepsilon)^\nu}{\nu!}\, x^{2j+\nu+\mu+1/2} \left| (x^{-1}D)^{n+j}\, x^{-\mu-1/2} (h_\mu \Phi(x) \right| \qquad \text{... (3.7)}$$

Assume that $0 \le \mu + 1/2 \le p$, where p is positive integer then

$$x^{\mu+1/2} \le (1+x)^{\mu+1/2} \le (1+x)^p,$$

and the right-hand side of the eq. (3.7) is bounded by

$$e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} |b_j| \frac{(m\varepsilon)^\nu}{\nu!} (1+x)^{2j+\nu+p} \left| (x^{-1}D)^{n+j}\, x^{-\mu-1/2} (h_\mu \Phi(x) \right|.$$

$$\le e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} |b_j| \frac{(m\varepsilon)^\nu}{\nu!} \sum_{k=0}^{2j+\nu+p} \binom{2j+\nu+p}{k} x^k$$

$$\left| (x^{-1}D)^{n+j}\, x^{-\mu-1/2} (h_\mu \Phi(x) \right|.$$

Using Zemanian's technique and eq. (1.3) and the Leibnitz type formula (2.7), the last expression can be estimated by

$$e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} |b_j| \frac{(m\varepsilon)^\nu}{\nu!} \sum_{k=0}^{2j+\nu+p} \binom{2j+\nu+p}{k}$$

$$\times \int_0^\infty y^{2\mu+2(n+j)+k+1} \left| (y^{-1}D)^k \left[y^{-\mu-1/2} \Phi(y) \right] \right.$$

$$\left. \times (xy)^{-(\mu+n+j)} J_{\mu+k+(n+j)}(xy) \right| dy$$

$$\leq e^{mc(\varepsilon)} \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} |b_j| \sum_{k=0}^{2j+\nu+p} \frac{(m\varepsilon)}{\nu!} \binom{2j+\nu+p}{k} Q_\mu \int_0^\infty y^{2\mu+2(n+j)+k+1}$$

$$\times \left| (y^{-1}D)^k \left[y^{-\mu-1/2} \int_0^\infty (yt)^{1/2} J_\mu(yt) a(y,t)(h_\mu \phi)(t) dt \right] \right| dy$$

$$\leq e^{mc(\varepsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{2j+\nu+p} |b_j| \frac{(m\varepsilon)^\nu}{\nu!} \binom{2j+\nu+p}{k} \int_0^\infty \int_0^\infty y^{2\mu+2(n+j)+k+1}$$

$$\times \sum_{s=0}^{k} \binom{k}{s} t^{2(k-s)} \left| (y^{-1}D)^s \left[t^{2\mu+1} a(y,t) \right] \right| \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt\, dy.$$

Suppose that N is a positive integer no less than $2\mu + 6n + p + 1$, then the above expression can be estimated by

$$e^{mc(\varepsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{2j+\nu+p} |b_j| \frac{(m\varepsilon)^\nu}{\nu!}$$

$$\times \binom{2j+\nu+p}{k} \int_0^\infty \int_0^\infty \sum_{s=0}^{k} \binom{k}{s} t^{2(k-s)} (1+y)^{N+\nu+2} \left| (y^{-1}D)^s \left[t^{2\mu+1} a(y,t) \right] \right|$$

$$\times \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| dt \frac{dy}{(1+y)^2}.$$

Using the inequality $(1 + y) \leq e^{-a/b} e^{\omega(y)/b}$ by property (c) and also inequality (2.6) the right-hand of the above expression can be bounded by

$$e^{mc(\varepsilon)} Q_\mu^2 \sum_{\nu=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{2j+\nu+p} \sum_{s=0}^{k} |b_j| \frac{(m\varepsilon)^\nu}{\nu!} \binom{2j+\nu+p}{k}$$

$$\times \binom{k}{s} t^{2(k-s)} \int_0^\infty \exp(-(N+\nu+2)a/b))$$

$$\times c(\sigma, (N+\nu+2)/b)\, C^s s!(1+t)^{s+2} e^{\sigma\omega(t)} \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| \frac{dt}{(1+t)^2}$$

$$\leq e^{mc(\varepsilon)} Q_\mu^2 \sum_{v=0}^{\infty} \sum_{j=0}^{n} \sum_{k=0}^{2j+v+p} \sum_{s=0}^{k} |b_j| \frac{(m\varepsilon)^v}{v!}$$

$$\times \binom{2j+v+p}{k}\binom{k}{s} \int_0^\infty \exp(-(N+v+2)a/b))$$

$$\times c(\sigma, (N+v+2)\, a/b)\, C^s s!(1+t)^{2k+2} e^{\sigma\omega(t)} \left| t^{-\mu-1/2} (h_\mu \phi)(t) \right| \frac{dt}{(1+t)^2}.$$

Using again the property $(1+x) \leq e^{-a/b} e^{\omega(x)/b}$ and writing $d(\sigma, v)$ for $c(\sigma, (N+v+2)/b)$, we can estimate the last expression by

$$e^{mc(\varepsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^{\infty} \frac{(m\varepsilon)^v}{v!} e^{-av/b} \sum_{j=0}^{n} |b_j| \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k}$$

$$\times \sup_{k \in [0, 2+4n+2v+2p]} \beta^\mu_{\sigma+(2k+2/b), 0} (h_\mu \phi)\, d(\sigma, v) \sum_{s=0}^{k} \binom{k}{s} C^s s!$$

$$\leq e^{mc(\varepsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^{\infty} \frac{(m\varepsilon)^v}{v!}$$

$$\times e^{-av/b} \sum_{j=0}^{n} |b_j| Q^\mu_{n, v, p} \sum_{k=0}^{2j+v+p} \binom{2j+v+p}{k} (1+C)^k k!$$

$$\leq e^{mc(\varepsilon)} Q_\mu^2 \exp[-a(N+2)/b]$$

$$\sum_{v=0}^{\infty} \left[m\varepsilon\, e^{-a/b} (Q^\mu_{n, v, p})^{1/v} \right]^v \sum_{j=0}^{n} |b_j| (2+c)^{2j+vp} \frac{(2j+v+p)!}{v!}, \qquad \ldots (3.8)$$

where $$Q^\mu_{n, v, p} = d(\sigma, v) \sup_{k \in [0, 2+4n+2v+2p]} \beta^\mu_{\sigma+(2k+2)/b, 0} (h_\mu \phi).$$

Setting $T = 2 + c$ and using the result $\Gamma(v+\alpha)/\Gamma(v+\beta) = O(v^{\alpha-\beta})$, $v \to \infty$, the expression (3.8) can also estimated by

$$A e^{mc(\varepsilon)} Q_\mu^2 \exp[-a(N+2)/b] \sum_{v=0}^{\infty} e^{(2n+p)\log v}$$

$$\times \exp\left[v \log\left(m\varepsilon T \left(Q^\mu_{n, v, p} \right)^{1/v} e^{-a/b} \right) \right] \sum_{j=0}^{n} |b_j| T^{2j+p}$$

for some positive constant A.

Since $\log v < v$ we can further estimate it as follows:

$$e^{mc(\varepsilon)} Q_{\mu}^{2} \exp\left[-a(N+2)/b\right] \times \sum_{v=0}^{\infty} \exp\left[v \left\{ \log\left(m\varepsilon T \left(Q_{n,v,p}^{\mu} \right)^{1/v} \right) - a/b + 2n + p \right\} \right] \times \sum_{j=0}^{n} |b_j| T^{2j+p} < \infty$$

as the infinite series can be made convergent by choosing

$$\varepsilon < (mT)^{-1} (Q_{n,v,p}^{\mu})^{-1/v} \exp(-(2n + p - a/b)).$$

This completes the proof of the theorem.

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