Indian J. pure appl. Math., 25(3); 337-343, March 1994

A GENERATING SOLUTION OF THE RESTRICTED THREE-BODY PROBLEM IN *KS*-VARIABLES

M. R. HASSAN

Department of Mathematics, Millat College, Parsa
K. Parsa, Godda, (Bihar)

(Received 25 *May* 1992; *after revision* 26 *July* 1993;
accepted 13 *August* 1993*)*

In this paper a new form of generating solution has been established in the restricted problem of three bodies in a four-dimensional phase space.

INTRODUCTION

Krasinski[2] established the generating solutions in the circular restricted problem of two bodies in planar case. Ahmad and Huda[1] generalized the problem in the circular restricted problem of three bodies in four-dimensional phase space dealing with some particular cases. For example they have chosen the inclination of the orbital plane as 90° and also they took 8 as the constant of integration of the energy integral without any reason. In this paper, we have presented a generating solution of the restricted three-body problem in terms of *KS*-variables.

Here all the orbital elements a, e, ω, J, Ω and E are introduced in the generalized form, where

a = semi-major axis of the elliptic orbit,

e = eccentricity of the elliptic orbit,

ω = argument of the perihelion,

J = inclination of the orbital plane of the infinitesimal mass with the equatorial plane,

Ω = longitude of the ascending node,

E = eccentric anomaly.

First five elements are constants and E is only the variable in this problem. *KS*-variables are used here also, which transforms three dimensional physical space into four-dimensional phase space.

Equations of Motion

If μ be the ratio of the mass of the smaller primary to the total mass of the primaries, then the regularized canonical equations of motion of the infinitesimal mass given by Kurcheeva[4], are

$$\frac{dq_i}{ds} = \frac{\partial K}{\partial Q_i} \qquad \dots (1)$$

$$\frac{dQ_i}{ds} = -\frac{\partial K}{\partial q_i} \qquad (i = 1, 2, 3, 4)$$

where q_i's are the generalized co-ordinates and Q_i's are the corresponding generalized components of momenta with the Hamiltonian K, given by

$$K = \frac{1}{2}\sum_{i=1}^{4} Q_i^2 + 2\rho^2 [\, Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3 \,] - 4(1-\mu)$$

$$+ 4\rho^2 [\, C/2 - \frac{\mu^2}{2} - \frac{\mu}{r_2} + \mu \sum_{i=1}^{4} (-1)^{i+1} q_i^2 \,].$$

The distance between the infinitesimal mass and the bigger primary is given by

$$r_1 = \sum_{i=1}^{4} q_i^2 = \rho^2. \qquad \dots (3)$$

The distance between the infinitesimal mass and the smaller primary is given by

$$r_2^2 = 1 - 2 \sum_{i=1}^{4} (-1)^{i+1} q_i^2 + \left[\sum_{i=1}^{4} q_i^2 \right]^2. \qquad \dots (4)$$

The physical time t and fictitious time s are connected by the relation

$$dt = \rho^2 \, ds.$$

For generating solution, $\mu = 0$, then from (2)

$$K = \frac{1}{2}\sum_{i=1}^{4} Q_i^2 + 2\rho^2 [\, Q_1 q_2 - Q_2 q_1 - Q_3 q_4 + Q_4 q_3 + C] - 4. \qquad \dots (6)$$

With the help of (1) and (6) the followings are obtained

$$\dot q_1 = Q_1 + 2\rho^2 q_2, \qquad Q_1 = \dot q_1 - 2\rho^2 q_2$$

$$\dot q_2 = Q_2 - 2\rho^2 q_1, \qquad Q_2 = \dot q_2 + 2\rho^2 q_1$$

$$\dot q_3 = Q_3 - 2\rho^2 q_4, \qquad Q_3 = \dot q_3 + 2\rho^2 q_4$$

$$\dot q_4 = Q_4 + 2\rho^2 q_3, \qquad Q_4 = \dot q_4 - 2\rho^2 q_3 \quad \left(\cdot = \frac{d}{ds} \right) \qquad \dots (7)$$

Again with the help of (1), (3), (6) and (7) one can easily show that

$$\ddot{q}_1 - 4\rho^2 \dot{q}_2 - 4(q_1^2 + q_2^2)\, \dot{q}_2 - 4(q_2 q_3 + q_1 q_4)\, \dot{q}_3 - 4(q_2 q_4 - q_1 q_3) \dot{q}_4$$

$$+ \; 4q_1 (3\rho^4 - C) \qquad \text{... (8)}$$

$$\ddot{q}_2 + 4\rho^2 \dot{q}_1 + 4(q_1^2 + q_2^2)\, \dot{q}_1 + 4(q_1 q_3 - q_2 q_4)\, \dot{q}_3 + 4(q_1 q_4 + q_2 q_3) \dot{q}_4$$

$$= 4q_2 (3\rho^4 - C) \qquad \text{... (9)}$$

$$\ddot{q}_3 - 4\rho^2_{q_4} + 4\,(q_1 q_4 + q_2)\, \dot{q}_1 + 4(q_2 q_4 - q_1 q_3)\, \dot{q}_2 + 4(q_3^2 + q_4^2)\, \dot{q}_4$$

$$= 4q_3 (3\rho^4 - C) \qquad \text{... (10)}$$

$$\ddot{q}_4 - 4\rho^2 \dot{q}_3 - 4(q_1 q_3 - q_2 q_4)\, \dot{q}_1 - 4(q_2 q_3 + q_1 q_4)\, \dot{q}_2 - 4(q_3^2 + q_4^2)\, \dot{q}_3$$

$$= 4q_4 (3\rho^4 - C). \qquad \text{... (11)}$$

Multiplying eqns. (8), (9), (10) and (11) by q_2, q_1, q_4 and q_3 respectively and subtracting the sum of 2nd and 3rd from the sum of the 1st and 4th and on integration, one can easily find that

$$\dot{q}_1 q_2 - q_1 \dot{q}_2 - \dot{q}_3 q_4 + q_3 \dot{q}_4 = 2\,\rho^4 + \lambda. \qquad \text{... (12)}$$

Again multiplying eqns. (8), (9), (10) and (11) by $\dot{q}_1, \dot{q}_2, \dot{q}_3$ and $\dot{q}_4$ respectively and integrating their sum, the result follows that,

$$\sum_{i=1}^{4} \dot{q}_i^2 = 4\,\rho^6 - 4C\,\rho^2 + \nu \qquad \text{... (13)}$$

where λ and ν are the constants of integration depending upon the initial conditions.

In terms of constant orbital elements, a, e, ω, J, Ω and the variable eccentric anomaly E, the explicit representations of KS-variables of an elliptic motion are given as (Stiefel and Scheifele[5]),

$$q_1 = \sqrt{a}\, \sin J/2\, [\, \sqrt{1-e}\, \cos\theta \cos E/2 + \sqrt{1+e}\, \sin\theta \sin E/2\,]$$

$$q_2 = \sqrt{a}\, \sin J/2\, [\, \sqrt{1-e}\, \sin\theta \cos E/2 - \sqrt{1+e}\, \cos\theta \sin E/2\,]$$

$$q_3 = \sqrt{a}\, \cos J/2\, [\, \sqrt{1-e}\, \sin\phi \cos E/2 + \sqrt{1+e}\, \cos\phi \sin E/2\,] \qquad \text{... (14)}$$

$$q_4 = \sqrt{a}\, \cos J/2\, [\, -\sqrt{-1-e}\, \cos\phi \cos E/2 + \sqrt{1+e}\, \sin\phi \sin E/2\,]$$

$$\left[\; \theta = \frac{\Omega - \omega}{2},\; \phi = \frac{\Omega + \omega}{2} \text{ are constants} \right]$$

which satisfy the relations

$$r_1 = \rho^2 = \sum_{i=1}^{4} q_i^2 = a(1 - e\cos E) \quad \text{... (15)}$$

$$r_2^2 = 1 - \sum_{i=1}^{4} (-1)^{i+1} q_i^2 + \left[\sum_{i=1}^{4} q_i^2 \right]^2$$

$$= 1 - 2a\,[A_1(\cos E - e) + A_2\sqrt{1-e^2}\sin E] + a^2(1 - e\cos E)^2 \quad \text{... (16)}$$

where $A_1 = \sin\Omega\sin\omega - \cos\Omega\cos\omega\cos J$

$$A_2 = \sin\Omega\cos\omega + \cos\Omega\sin\omega\cos J \quad \text{... (17)}$$

and $dt = \rho^2\,ds = a(1 - e\cos E)\,ds.$ ... (18)

Differentiating q_i's with respect to s,

$$\dot{q}_1 = \frac{1}{2}\sqrt{a}\sin J/2\ [-\sqrt{1-e}\cos\theta\ \sin E/2 + \sqrt{1+e}\ \sin\theta\cos E/2]\,\dot{E}$$

$$\dot{q}_2 = \frac{1}{2}\sqrt{a}\sin J/2\ [-\sqrt{1-e}\sin\theta\ \sin E/2 - \sqrt{1+e}\ \cos\theta\cos E/2]\,\dot{E}$$

$$\dot{q}_3 = \frac{1}{2}\sqrt{a}\cos J/2\ [-\sqrt{1-e}\sin\phi\ \sin E/2 + \sqrt{1+e}\ \cos\phi\cos E/2]\,\dot{E}$$

$$\dot{q}_4 = \frac{1}{2}\sqrt{a}\cos J/2\ [\sqrt{1-e}\cos\phi\ \sin E/2 + \sqrt{1+e}\ \sin\phi\cos E/2]\,\dot{E}$$

... (19)

which satisfy the relations

$$\dot{q}_1 q_2 - q_1\dot{q}_2 - q_3 q_4 + q_3\dot{q}_4 = \frac{a}{2}\sqrt{1-e^2}\ \dot{E} \quad \text{... (20)}$$

and $$\sum_{i=1}^{4} \dot{q}_i^2 = \frac{a}{4}(1 + e\cos E)\,\dot{E}^2. \quad \text{... (21)}$$

The results obtained from the combination of eqns. {(12), (!5), (20)} and {(13), (15), (21)} are

$$\frac{a}{2}\sqrt{1-e^2}\ \dot{E} = 2a^2(1 - e\cos E)^2 + \lambda \quad \text{... (22)}$$

and $$\frac{a}{4}(1 + e\cos E)\,\dot{E}^2 = 4a^3(1 - e\cos E)^3 - 4aC(1 - e\cos E) + \nu. \quad \text{... (23)}$$

But in an elliptic orbit, the eccentric anomaly, E is given by

$$E = 2\omega s \quad (\omega = \sqrt{h/2}) \quad \text{... (24)}$$

where h is the total energy of the Kepler motion and ω is the angular frequency.

From (18)

$$dt = \rho^2\, ds = (1 - e \cos E)\, ds = (1 - e \cos 2\,\omega\, s)\, ds.$$

On integration

$$t = \frac{a}{2\omega}\,(E - e \sin E). \qquad \ldots (25)$$

In terms of stumpff-functions and fictitious time s, the physical time t can be written as

$$t = \int \rho^2\, ds = C_0 + C_1\, s + \frac{C_2}{2\omega} \sin 2\omega\, s - \frac{C_3}{2\omega} \cos 2\omega\, s \qquad \ldots (26)$$

where C_0 is the constant of integration and C_1, C_2, C_3 are the stumpff-functions given by

$$C_1 = \frac{|\vec{\alpha}|^2 + |\vec{\beta}|^2}{2} = a \text{ semi-major axis,}$$

$$C_2 = \frac{|\vec{\alpha}|^2 - |\vec{\beta}|^2}{2} = -\,ae, \qquad \text{([5], pp-39)}$$

and $\quad C_3 = (\vec{\alpha}\,, \vec{\beta}\,).$

Here $\vec{\alpha}$, $\vec{\beta}$ are the vectorial constants involving in the general solution of the pure Kepler motion.

Since for the unperturbed Kepler motion

$$\frac{d\vec{\alpha}}{dE} = 0,\ \frac{d\vec{\beta}}{dE} = 0 \text{ hence } \frac{dC_3}{ds} = 0 \text{ [as } (E = 2\omega\, s)].$$

Now combining eqns. (25) and (26) we get

$$\frac{a}{2\omega}\,(E - e \sin E) = C_0 + C_1\, s + \frac{C_2}{2\omega} \sin 2\omega\, s - \frac{C_3}{2\omega} \cos 2\omega\, s. \qquad \ldots (27)$$

Differentiating this equation and following the initial conditions

$E = 0$ at $s = 0$ we get $\acute{E} = 2\omega$. ... (28)

Now we proceed for the constants of integration λ and ν in (22) and (23).

Initially at the pericentre passage, $E = 0$, $\acute{E} = 2\omega$, then from (22)

$$\lambda = a\omega \sqrt{1 - e^2} - 2a^2\,(1 - e)^2.$$

Therefore eqn. (22) reduces to

$$\frac{a}{2}\sqrt{1 - e^2}\ \acute{E} = 2a^2\,(1 - e \cos E)^2 + a\omega \sqrt{1 - e^2} - 2a^2\,(1 - e)^2.$$

For small value of eccentricity neglecting the second and higher order terms of

e, the final form of the above equation is

$$\dot{E} = x - y\cos E \quad \text{... (29)}$$

where $x = 2\omega + 8ae, y = 8ae$ are constants.

Here clearly $x > y$, then eqn. (29) yields

$$E = 2\tan^{-1}\left[p\tan\left(\frac{s}{2}\sqrt{x^2 - y^2} + pp_0\right)\right] \quad \text{... (30)}$$

where $p = \sqrt{\dfrac{x-y}{x+y}}$ and p_0 is the constant of integration.

Again initially at the pericentre passage, $E = 0$, $\dot{E} = 2\omega$, then from (23) we get

$$v = a\omega^2(1+e) - 4a^3(1-e) + 4aC(1-e).$$

Therefore eqn. (23) reduces to

$$\frac{a}{4}(1 + e\cos E)\dot{E}^2 = 4a^3(1 - e\cos E)^3 - 4aC(1 - e\cos E) + a\omega^2(1+e)$$

$$- 4a^3(1-e)^3 + 4ac(1-e).$$

For small value of eccentricity, neglecting the second and higher order terms of e, the final form of the above equation is

$$\dot{E} = \sqrt{l - m\cos E}, \quad \text{... (31)}$$

where $l = 4\omega^2 + 4a(\omega^2 + 12a^2 - 4c)$, $m = 4e(\omega^2 + 12a^2 - 4C)$ are constants.

Here $l \neq m$ and hence analytical integration is not possible.

The solution (30) is not found suitable for showing the periodic solution of the motion of the infinitesimal mass. Therefore, it may be possible that there will be some other integrals, which will produce the periodic solutions.

With the help of (14) and (19), one integral has been found as

$$q_1\dot{q}_4 - q_2\dot{q}_3 + \dot{q}_2 q_3 - \dot{q}_1 q_4 = 0. \quad \text{... (32)}$$

Without numerical verification it can not be justified that which one will produce periodic solution.

The relation between the physical time t and fictitious time s can be given as

$$t = \int a(1 - e\cos E)\,ds. \quad \text{... (33)}$$

On substituting the values of E from (30) in (33), one can integrate numerically, as analytical method is not suitable for (33)

Therefore the generating solutions are

$$q_1 = \gamma\left[\sqrt{1-e}\cos\theta\cos E/2 + \sqrt{1+e}\sin\theta\sin E/2\right]$$

$$q_2 = \gamma\left[\sqrt{1-e}\sin\theta\cos E/2 + \sqrt{1+e}\cos\theta\sin E/2\right]$$

$$q_3 = \delta\,[\,\sqrt{1-e}\,\sin\phi\cos E/2 + \sqrt{1+e}\,\cos\phi\sin E/2\,]$$

$$q_4 = \delta\,[\,\sqrt{1-e}\,\cos\phi\cos E/2 + \sqrt{1+e}\,\sin\phi\sin E/2\,] \qquad \ldots (34)$$

$$\gamma = \sqrt{a}\,\sin J/2, \quad \delta = \sqrt{a}\,\cos J/2$$

$$Q_1 = \frac{\gamma \dot{E}}{2}\,[-\sqrt{1-e}\,\cos\theta\sin E/2 + \sqrt{1+e}\,\sin\theta\cos E/2\,]$$

$$-\;2\gamma a\,(1-e\cos E)\,[\,\sqrt{1-e}\,\sin\theta\cos E/2 - \sqrt{1+e}\,\cos\theta\sin E/2]$$

$$Q_2 = \frac{\gamma \dot{E}}{2}\,[-\sqrt{1-e}\,\sin\theta\sin E/2 - \sqrt{1} + e\cos\theta\cos E/2\,]$$

$$+\;2\gamma a\,(1-e\cos E)\,[\,\sqrt{1-e}\,\cos\theta\cos E/2 + \sqrt{1+e}\,\sin\theta\sin E/2]$$

$$Q_3 = \frac{\delta \dot{E}}{2}\,[-\sqrt{1-e}\,\sin\phi\sin E/2 + \sqrt{1+e}\,\cos\phi\cos E/2\,]$$

$$+\;2\delta a\,(1-e\cos E)\,[-\sqrt{1-e}\,\cos\phi\cos E/2 + \sqrt{1+e}\,\sin\phi\sin E/2]$$

$$Q_4 = \frac{\delta \dot{E}}{2}\,[\,\sqrt{1-e}\,\cos\phi\sin E/2 + \sqrt{1+e}\,\sin\phi\cos E/2\,]$$

$$-\;2\delta a\,(1-e\cos E)\,[\,\sqrt{1-e}\,\sin\phi.\cos E/2 + \sqrt{1+e}\,\cos\phi\sin E/2] \qquad \ldots (35)$$

where $\dot{E} = x - y\cos E$ and $\dot{E} = \sqrt{l - m\cos E}, x, y, l$ and m have their usual meaning as in (29) and (31).

ACKNOWLEDGEMENT

The author expreses deep sense of gratitude to Prof. K. B. Bhatnagar of Delhi University for his valuable suggestions and helping attitude to revise the paper after referee's comments and also I am grateful to the referee who gave me some important directions to rectify the paper.

REFERENCES

1. A. Ahmad and M.N. Huda, *Bull. Astr. Soc. India*, **14** (1986), 11.
2. G. A. Krasinski, *Bull. Theor. Astron.* **9** (1963), 105.
3. P. Kustaanheimo and E. L. Stiefel *J. Reine-Angew, Math.* **218** (1965), 204.
4. I. V. Kurcheeva, *Cel. Mech.* **15** (1977), 353.
5. E. L. Stiefel and G. Scheifele, *Linear and Regular Celestial Mechanics*, Academic Press, New York, 1971.