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ON STATISTICALLY CONVERGENT AND STATISTICALLY CAUCHY SEQUENCES

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The concept of Statistically cauchy Sequences on a Hausdorff locally convex topological vector space is introduced and a decomposition theorem for a Statistically Cauchy sequence is established. A necessary and sufficient condition for a Statistically Cauchy sequence to be a Cauchy Sequence is obtained. Finally, some equivalent critiera for the sequential completeness of a Hausdorff l.c.t.v.s related to statistical convergence are established.

1. PRELIMINARIES

The notion of statistical convergence was introduced by Fast[3] and also independently by Buck[1] and Schoenberg[7] for real and complex sequences. Maddox[5] extended this concept for sequences in any Hausdorff locally convex topological vector spaces (Hausdorff l.c.t.v.s. in brief). Fridy[4] obtained an equivalent criterion for statistically convergent real sequences, similar to the Cauchy criterion of convergence. One of our objects here is to study statistically Cauchy sequences in a Hausdorff l.c.t..s. X and establish a decompostion theorem for such sequences. A Tauberian result for any statistically Cauchy sequence to be statistically convergent is proved which extends a result due to Maddox. Finally, equivalent conditions are investigated for the sequential completeness of X. The generating family of seminorms for X is denoted by Q.

A subset A of the ordered set N of natural numbers is said to have density $\delta(A)$, if $\lim \frac{|A(n)|}{n} = \delta(A)$, where $A(n) = \{k < n : k \in A\}$ and $|A|$ denotes the cardinality of any set $A \subset N$. Clearly, finite sets have zero density and $\delta(A') = 1 - \delta(A)$, whenever either side exists and $A' = N - A$. If a property $P(k)$ holds for all $k \in A$ with $\delta(A) = 1$, we say that P holds for almost all k, that is a.a.k.

A sequence $x = (x_k)$ in X is said to be statistically convergent to l Maddox[5] if for any $q \in Q$ and $\varepsilon > 0$.

$$\delta(\{k \in N : q(x_k - l)) \geq \varepsilon\}) = 0.$$

In symbols, we write $x_k \overset{\text{stat}}{\rightarrow} l$. We say that x is a statistically Cauchy sequence if for any $q \in Q$ and $\varepsilon > 0$, there exists $m = m(\varepsilon, q) \in N$ such that $\delta(\{k \in N : q(x_k - x_m) \geq \varepsilon\}) = 0$. Every statistically convergent sequence is statistically Cauchy. Following Maddox[5], we say that (x_k) is slowly oscillating if for each $q \in Q$, $q(x_n - x_k) \rightarrow 0$ as $k \rightarrow \infty$, whenever $n/k \rightarrow 1$ with $n \geq k$. A Cauchy sequence is slowly oscillating, but not conversely. The symbols $\omega(x)$, $\bar{c}(X)$ and $\bar{c}_0(X)$ denote respectively the spaces of all sequences, all statistically convergent sequences and sequences that are statistically convergent to the zero element θ of X. For any $x = (x_k) \in \omega(X)$, supp $x' = \{k \in N : x_k \neq \theta\}$.

2. The Main Results

In Theorem 1, some equivalent conditions for statistical convergence are listed, whose analogues for complex sequences are wellknown[2,4,6].

Theorem 1 — If $x = (x_k) \in \omega(X)$, then the following conditions are equivalent.

(a) $x_k \overset{\text{stat}}{\rightarrow} l$.

(b) For any $q \in Q$, there exist $y = (y_k)$ and $z = (z_k)$ in $\omega(X)$ such that $x = y + z$, $q(y_k - l) \rightarrow 0$ and δ (supp z) $= 0$, $l \in X$.

(c) For any $q \in Q$, there is a subsequence $K = \{k_n\}$ of N such that $\delta(K) = 1$ and $q(x_{k_n} - l) \rightarrow 0$, $l \in X$.

Proof : The proof that (a) implies (b) is similar to that of Theorem 2.3 in Connor[2], with q in place of modulus. Suppose that (b) holds and define $K = (k_n)$ to be the subsequence of N such that $k \in K$ if and only if $z_k = \theta$. Then $\delta(K) = \delta(N - \text{supp } z) = 1$ since $\delta(\text{supp } z) = 0$ by assumption. Since $x_{k_n} = y_{k_n}$ for each $n \in N$ and $q(y_k - l) \rightarrow 0$, it follows that $q(x_{k_n} - l) \rightarrow 0$, so that (c) holds. Thus (b) implies (c). Finally, if (c) is assumed, then for any $q \in Q$ and $\varepsilon > 0$, there exists a subsequence $K = (k_n)$ of N with $\delta(K) = 1$ and $m = m(\varepsilon, q) \in N$ such that $q(x_{k_n} - l) < \varepsilon$ for all $n \geq m$. Then $\delta(\{k \in N : q(x_k - l) \geq \varepsilon)\} \leq \delta(N - \{k_n : n > m\})$

$$= 1 - \delta(\{k_n : n > m\}) = 0$$

implying that $x_k \overset{\text{stat}}{\rightarrow} l$. Thus (c) implies (a) and the proof is complete.

In the next result, some equivalent alternative conditions are obtained for a sequence to be statistically Cauchy.

Theorem 2 — For any $x = (x_k) \in \omega(X)$, the following are equivalent.

(a) x is a statistically Cauchy sequence.

(b) For any $q \in Q$, there exists a subsequence $K = (k_n)$ of N with $\delta(K) = 1$ such that $q(x_{k_m} - x_{k_n}) \rightarrow 0$ as $m, n \rightarrow \infty$.

(c) For any $q \in Q$, there exist $y = (y_k)$ and $z = (z_k)$ in $\omega(X)$ such that $x = y + z$, $q(y_m - y_n) \to 0$ as $m, n \to \infty$ and δ (supp z) = 0.

PROOF : To prove that (a) implies (b), let $x = (x_k)$ be a statistically Cauchy sequence. By definition, we can find $m_1 \in N$ for any $q \in Q$ such that if k_1 is the set $\{k \in N : q(x_{m_1} - x_k) < \frac{1}{2}\}$, then $\delta(K_1) = 1$. Similarly, we can find $m_2 \in N$ such that $\delta(B_1) = 1$, where $B_1 = \{k \in N : q(x_k - x_{m_2}) < 1/4\}$. If we set $K_2 = K_1 \bigcap B_1$, then $\delta(K_2) = 1$, $K_2 \subset K_1$ and $q(x_{k_1} - x_{k_2}) < \frac{1}{2}$ for all $k_1, k_2 \in K_2$. Proceeding inductively, we can construct a decreasing sequence (K_j) of subsets of N such that

$$\delta(K_j) = 1 \text{ for each } j \in N, \qquad \text{... (2.1)}$$

and

$$q(x_{k_1} - x_{k_2}) \le \frac{1}{j} \text{ for all } k_1, k_2 \in K_j,\ j \in N. \qquad \text{... (2.2)}$$

Next we construct a subsequence (v_j) of N as follows. Let $v_1 \in K_1$. By (2.1), we can find $v_2 \in K_2$ with $v_2 > v_1$ such that for each $n \ge v_2$, $\frac{|K_2(n)|}{n} > \frac{1}{2}$. The subsequence (v_j) can be defined inductively such that $v_j \in K_j$ for each $j \in N$ and

$$\frac{|K_j(n)|}{n} > \frac{j-1}{j}, \text{ for each } n \ge v_j. \qquad \text{... (2.3)}$$

If we set

$$K = \{k \in N : 1 \le k < v_1\} \bigcup \Big[\bigcup_{j \in N} \{k : v_j \le k < v_{j+1}\} \bigcap K_j \Big]$$

then for any $j \in N$ and $v_j \le n < v_{j+1}$,

$$K_j(n) = \{k \le n : k \in K_j\} \subset \{k \le n : k \in K\} = K(n),$$

which implies by (2.3) that for all $j \in N$,

$$\frac{|K(n)|}{n} \ge \frac{|K_j(n)|}{n} > \frac{j-1}{j},$$

and hence $\delta(K) = 1$.

Now we show that $q(x_m - x_n) \to 0$ as $m, n \to \infty$ and $m, n \in K$. For this, let $\varepsilon > 0$ and let $j \in N$ be such that $j > \frac{1}{\varepsilon}$. If $m, n \in K$ and $m, n > v_j$, we can find $r, s \ge j$ such that $v_r \le m < v_{r+1}$, $v_s \le n < v_{s+1}$, so that $m \in K_r$, $n \in K_s$. Suppose that $r \le s$. Then $K_s \subset K_r$, so that $m, n \in K_r$. By (2.2),

$$q(x_m - x_n) \le \frac{1}{r} \le \frac{1}{j} < \varepsilon$$

which proves our assertion. Thus (a) implies (b).

Next, let (b) hold. Then for a given $q \in Q$, there is $K = (k_n)$ such that $\delta(K) = 1$ and $q(x_{k_n} - x_{k_m}) \to 0$ as $m, n \to \infty$. Define

$$y_k = x_k,\ z_k = \theta \quad \text{if} \quad 1 \le k \le k_1 \text{ and}$$

$$y_k = x_{k_n},\ z_k = x_k - x_{k_n} \quad \text{if } k_n \le k < k_{n+1}, \quad n \in N.$$

Then $x = y + z$, δ (supp z) $\le \delta(K') = 0$ and construction shows that $q(y_{k_n} - y_{k_m}) \to 0$ as $m, n \to \infty$. This proves (c).

Lastly, if (c) is assumed to hold and for any $q \in Q$, y and z are as determined there, then $x_k = y_k$ for all $k \in N = N$ − supp z = (k_n) with $\delta(K) = 1$. For a given $\varepsilon > 0$, let $m \in N$ be such that

$$q(x_{k_m} - x_{k_n}) < \varepsilon \text{ for all } n \ge m.$$

Then

$$\delta(\{k \in N : q(x_k - x_{k_m}) \ge \varepsilon\}) \le \delta(N - \{k_{m+1}, k_{m+2}, \ldots\}) = 0,$$

so that (a) holds. This completes the proof of the theorem.

The following Tauberian theorem can be established now.

Theorem 3 — In a Hausdorff l.c.t.v.s. X, if $x \in \omega(x)$ is statistically Cauchy, then x is a Cauchy sequence if and only if it is slowly oscillating.

PROOF : The necessity is trivial. To prove sufficiency, let $x = (x_k)$ be a statistically Cauchy sequence in X which is slowly oscillating. For a given $q \in Q$, choose $K = (k_n)$ as in Theorem 2(b). Since $\delta(K) = 1$, $\lim\limits_{n \to \infty} \dfrac{|\{k \le k_n : k K\}|}{k_n} = \lim \dfrac{n}{k_n} = 1$. This implies, since x is slowly oscillating, that $q(x_n - x_{k_n}) \to 0$, as $n \to \infty$ and hence for $m, n \in N$, $q(x_n - x_m) \le q(x_n - x_{k_n}) + q(x_{k_n} - x_{k_m}) + q(x_{k_m} - x_m)$

$$\to 0, \text{ as } m,\ n \to \infty.$$

This completes the proof of the theorem.

Corollary (Maddox[5], Theorem 3) — If $x_k \xrightarrow{\text{stat}} l$ and (x_k) is slowly oscillating, then (x_k) converges to l.

PROOF : By Theorem 3, (x_k) is a Cauchy sequence in X. Hence, for each $q \in Q$, $q(x_m - x_n) \to 0$ as $m, n \to \infty$. By Theorem 1(c), there is a subsequence (k_n) of N such that $q(x_{k_n} - l) \to 0$. It follows that $q(x_n - l) \to 0$ as $n \to \infty$ as and the corollary is proved.

3. SEQUENTIAL COMPLETENESS AND STATISTICAL CONVERGENCE

For any Hausdorff l.c.t.v.s. X with a generating family Q of seminorms, we define $l_\infty(X)$ to be the subspace of $\omega(X)$ consisting of elements $x = (x_k)$ such that

$\sup_k q(x_k) < \infty$ for each $q \in Q$. The family $\overline{Q} = \{\overline{q} : q \in Q\}$ of seminorms on $l_\infty(X)$, where $\overline{q}(x) = \sup_k q(x_k)$ for each $x = (x_k) \in l_\infty(X)$, make it a Hausdorff l.c.t.v.s. It is routine work to check that $l_\infty(X)$ is sequentially complete if and only if X is. We now prove the following theorem.

Theorem 4 — For any Hausdorff l.c.t.v.s. X, the following are equivalent.

(a) X is sequentially complete.

(b) $\overline{c}(X) \bigcap l_\infty(X)$ is sequentially complete in $l_\infty(X)$.

(c) Every sequence of the form $y + z$, where y is Cauchy in X and δ (supp z) $= 0$, is statistically convergent.

PROOF : Assume (a) and let $(x^{(n)})$ be a Cauchy sequence in $\overline{c}(X) \bigcap l_\infty(X)$, where $x^{(n)} = (x_k^{(n)})$ and $x_k^{(n)} \overset{\text{stat}}{\to} a_n$ for each $n \in N$, Since $l_\infty(X)$ is sequentially complete whenever X is, $\lim_{n\to\infty} x^{(n)} = x = (x_k)$ exists in $l_\infty(X)$. We show that $x \in \overline{c}(X)$.

First, it is shown that $\lim_{n\to\infty} a_n$ exists in X. Since $(x^{(n)})$ is a Cauchy sequence, choose $n_0 \in N$ for any given $q \in Q$ and $\varepsilon > 0$ such that

$$\overline{q}(x_k^{(m)} - x_k^{(n)}) < \frac{\varepsilon}{3}, \; m, n \geq n_0. \quad \text{... (3.1)}$$

For fixed $q \in Q$ and $m, n \in N$, we have

$$q(x_k^{(m)} - a_m) < \frac{\varepsilon}{3} \text{ and } q(x_k^{(n)} - a_n) < \frac{\varepsilon}{3} \text{ a.a.k,} \quad \text{... (3.2)}$$

and hence for any m, $n \geq n_0$, we can choose $k \in N$ such that (3.2) holds. Then

$$q(a_m - a_n) \leq q(a_m - x_k^{(m)}) + q(a_n - x_k^{(n)}) + q(x_k^{(m)} - x_k^{(n)})$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon$$

by (3.1) and (3.2), so that (a_n) is a Cauchy sequence in X. By hypothesis (a), $\lim_{n\to\infty} a_n = a \in X$..

Next, we show that $x_k \overset{\text{stat}}{\to} a$, or equivalently, for any given $\eta > 0$ and $q \in Q$,

$$q(x_k - a) < \eta \text{ a.a.k.} \quad \text{... (3.3)}$$

Choose $m \in N$ such that

$$q(a_m - a) < \frac{\eta}{3}, \; \overline{q}(x^{(m)} - x) < \frac{\eta}{3}. \quad \text{... (3.4)}$$

Since $x_k^{(m)} \overset{\text{stat}}{\to} a_m$, there is a subsequence K of N with $\delta(K) = 1$ such that

$$q\,(x_k^{(m)} - a_m) < \frac{\eta}{3}\ ,\ k \in K. \qquad \text{... (3.5)}$$

Then, for each $k \in K$, we have

$$q(x_k - a) \leq q\,(x_k - x_k^{(m)}) + q\,(x_k^{(m)} - a_m) + q\,(a_m - a)$$

$$< \frac{\eta}{3} + \frac{\eta}{3} + \frac{\eta}{3} = \eta,$$

by (3.4) and (3.5), which proves (3.3). Thus, (a) implies (b).

To prove that (b) implies (a), assume (b) and let (x_n) be any Cauchy sequence in X. If $x^{(n)}$ denotes the sequence $(x_n, x_n, x_n, ...)$ for $n \in N$, then for any $q \in Q$,

$$\bar{q}\,(x^{(m)} - x^{(n)}) = q(x_m - x_n) \to 0,\ \text{as}\ m, n \to \infty,$$

so that $(x^{(n)})$ is a Cauchy sequence in $\bar{c}\,(X) \bigcap l_\infty(X)$ and hence, by hypothesis (b), $\lim_{n \to \infty} x^{(n)}$ exists in $\bar{c}\,(X) \bigcap l_\infty(X)$. Since each $x^{(n)}$ is a constant sequence, so is $\lim_{n \to \infty} x^{(n)} = (a,\ a,\ a\ ...)$. Hence $\lim_{n \to \infty} x_{(n)} = a$ and (a) follows.

The equivalence of (a) and (c) is immediate in view of Theorem 1(b) and the fact that a Cauchy sequence which is also statistically convergent, is convergent. The proof of the Theorem 4 is complete.

Remark : It is easy to check that any one of the statements (a), (b) or (c) of Theorem 4 is implied by the following :

(d) Every statistically Cauchy sequence in X is statistically convergent.

The implication of (d) from (c), however, does not seem likely.

It is possible to show that X is complete if and only if $l_\infty(X)$ is complete and also that $\bar{c}\,(X) \bigcap l_\infty(X)$ is complete if and only if X is complete. The details are omitted.

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