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ON CERTAIN SPECIAL TRANSFORMATIONS OF POLY-BASIC ANALOGUE OF SRIVASTAVA-DAOUST'S FUNCTION

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Certain transformations of poly-basic hypergeometric function of Srivastava-Daoust have been established by the application of fractional q-derivative.

1. Introduction

In a recent paper Denis[5] has obtained certain transformations for the poly-basic hypergeometric function. In this paper, certain transformations of poly-basic hypergeometric function of Srivastava-Daoust have been established by the application of fractional q-derivative.

The fractional q-derivative of a product of two functions is defined as (cf. Agarwal[1]),

$$D_q^{\lambda}(UV) = \sum_{n\geq 0} \frac{(-1)^n q^{n(n+1)/2} (q^{-\lambda}; q)_n}{(q;q)_n} D_q^{\lambda-n} [U(xq^n)] D_q^n (V) \qquad \ldots..(1.1)$$

valid for $|x| < R$, $[R = \min(R_1, R_2)]$

where

$$U(x) = \sum_{r\geq 0} a_r x^r, \; |x| < R_1$$

$$V(x) = \sum_{r\geq 0} b_r x^r, \; |x| < R_2$$

and also

$$D_q^{\alpha} x^{u-1} = (1-q)^{-\alpha} \prod \begin{bmatrix} uq^{-\alpha}; \\ u \end{bmatrix} x^{u-\alpha-1}. \qquad \ldots(1.2)$$

2. Notations and Definitions

For $|q| < 1$, the q-shifted factorials are defined by

$$(a;\, q)_n = \begin{cases} 1, \text{ if } n = 0 \\ (1-a)\,(1-aq)\,\ldots\,(1-aq^{n-1}), \text{ if } n = 1, 2, \ldots, \end{cases}$$

with $\quad (a;\, q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n).$

Further

$$\prod \begin{bmatrix} a; \\ b \end{bmatrix} \text{ stands for } \prod_{n=0}^{\infty} \frac{(1-aq^n)}{(1-bq^n)}.$$

A poly-basic analogue of Srivastava and Daoust's[10] function is defined as

$$\Phi^{A:B+E}_{C:D+F}\begin{pmatrix} p : [(a:\theta)],\ q : [(b:\alpha)],\ t = [e:\gamma)] : x \\ p : [(c:\delta)],\ q : [(d:\beta)],\ t : [f:\epsilon)] \end{pmatrix}$$

$$= \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{A} (a_i,\, p)_{n\theta_i} \prod_{i=1}^{B} (b_1;\, q)_{n\alpha_i} \prod_{i=1}^{E} (e_i;\, t)_{n\nu_i}\, x^n}{\prod_{i=1}^{C} (c_i;\, p)_{n\delta_i} \prod_{i=1}^{D} (d_i;\, q)_{n\beta_i} \prod_{i=1}^{F} (f_i;\, t)_{n\epsilon_i}} \qquad \ldots(2.1)$$

where $|q| < 1$, $|p|$, $|t|$, $|x| < 1$, the argument x, the complex parameters

$$\begin{cases} a_i,\ i = 1,\ldots, A;\ b_i,\ i = 1,\ldots, B_i;\ e_i,\ i = 1,\ldots,E_i \\ c_i,\ i = 1,\ldots, C;\ d_i,\ i = 1,\ldots, D_i\, f_i,\ i = 1,\ldots,F_i \end{cases} \qquad \ldots(2.2)$$

and the non-negative real coefficients

$$\begin{cases} \theta_i,\ i = 1,\ldots, A;\ \alpha_i,\ i = 1,\ldots,B_i;\ \gamma_i,\ i = 1,\ldots,E_i \\ \delta_i,\ i = 1,\ldots,C\,;\ \beta_i,\ i = 1,\ldots,D_i;\ \epsilon_i,\ i = 1,\ldots,F_i \end{cases} \qquad \ldots(2.3)$$

being so constrained that the multiple series (2.1) converges. Where the parameters of the type (a_r) stand for the sequence of parameters $a_1,\ldots,a_r$ and will be denoted by (a), when $r = A$. It is evident that (2.1) defines a uni-basic hypergeometric function when $p = q = t$.

We will need the following basic analogue of Srivastava and Daoust's[10] function of two variables, which is a sepecial case for $n = 2$ of the corresponding n-variable definition due to Srivastava[9]

$$\Phi^{A:B;B'}_{C:D;D'}\begin{pmatrix} [(a:\theta,\phi)] : [(b:\alpha)]\,;\ [(b':\alpha')]\,; \\ [(c:\delta,\epsilon)] : [(d:\beta)]\,;\ [(d':\beta')]\,; \end{pmatrix} q;\, x,\, y \Bigg)$$

$$= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{\prod_{i=1}^{A} (a_i,\, q)_{m\theta_i + n\phi_i}}{\prod_{i=1}^{C} (c_1;\, q)_{m\delta_i + n\epsilon_i}}$$

$$\times \frac{\prod_{i=1}^{B} (b_i; q)_{m\alpha_i} \prod_{i=1}^{B'} (b'_i; q)_{n\alpha'_i}}{\prod_{i=1}^{D} (d_i; q)_{m\beta_i} \prod_{i=1}^{D'} (d'_i; q)_{n\beta'_i}} \frac{x^m}{(q; q)_m} \frac{y^n}{(q; q)_n}$$

$$|x|,\ |y|,\ |q| < 1. \qquad \ldots(2.4)$$

In what follows, the other notations will carry their usual meaning.

The results proved in this paper are of general character and include as special cases the results given earlier by Denis[5].

3. Transformations

In this section we establish the following transformation of poly-basic analogue of hypergeometric functions :

$$\Phi^{G:E}_{H:F}\left(\begin{matrix} q : [(g; \zeta)],\ p: [(e : \gamma')] \ ;\ z\xi \\ q : [(h, \rho)],\ p: [(f : \epsilon)] \ ; \end{matrix}\right)$$

$$\times\ \Phi^{A:C}_{B:D}\left(\begin{matrix} q : [(a : \theta)],\ t : [(c : \delta)] : y\eta \\ q : [(b : \alpha)],\ t : [(d : \beta)] \end{matrix}\right)$$

$$= \Phi^{A:E}_{B:F}\left(\begin{matrix} q : [(a; \theta)],\ p: [(e : \gamma')] \ ;\ y\xi \\ q : [(b, \alpha)],\ p: [(f : \epsilon)] \ ; \end{matrix}\right)$$

$$\times\ \Phi^{G:C}_{H:D}\left(\begin{matrix} q : [(g : \zeta)],\ t : [(c : \delta)] : z\eta \\ q : [(h : \rho)],\ t : [(d : \beta)] \ ; \end{matrix}\right). \qquad \ldots(3.1)$$

To prove (3.1), we make use of (1.1) and (1.2), by taking

$$U(x) = x^{\gamma-1}\ {}_G\Phi_H\left(\begin{matrix} [(g : \zeta)] \ ; \\ [(h : \rho)] \ ; \end{matrix}\ q \ ;\ xz\right)$$

and

$$V(x) = {}_A\Phi_B\left(\begin{matrix} [(a : \theta)] \ ; \\ [(b : \alpha)] \ ; \end{matrix}\ q \ ;\ xy\right)$$

to obtain

$$\sum_{n \geq 0} \frac{(-1)^n q^{n(n+1)/2} (q^{-\lambda} ; q)_n\, q^{n(\gamma-1)} x^n y^n \prod_{i=1}^{A} (a_i ; q)_{n\theta_i}}{(q ; q)_n (\gamma q^{-\lambda} ; q)_n \prod_{i=1}^{B} (b_i : q)_{n\alpha_i}}$$

$$\times\ {}_{G+1}\Phi_{H+1}\left(\begin{matrix} [(g : \zeta)],\ \gamma \ ; \\ [(h : \rho)],\ \gamma q^{n-\lambda} \ ; \end{matrix}\ q \ ;\ xzq^n\right)$$

$$\times \; {}_{A}\Phi_{B}\left(\begin{matrix}[(a)q^{n} : \theta]; \\ [(b)q^{n} : \alpha];\end{matrix}\; q;\, xy\right)$$

$$= \Phi^{1\,:\,G\,;\,A}_{1\,:\,H\,;\,B}\left(\begin{matrix}\gamma : [(g : \zeta)] \;;\; [(a : \theta)] \;; \\ \gamma q^{-\lambda} : [(h : \rho)] \;;\; [(b : \alpha)] \;.\end{matrix}\; q \;;\, xz,\, xy\right) \qquad \text{...(3.2)}$$

valid for $|xz| < 1$, $|xy| < 1$, $|q| < 1$.

Now, if we take

$$U(x) = x^{\gamma-1}\, {}_{A}\Phi_{B}\left(\begin{matrix}[(a : \theta)]; \\ [(b : \alpha)]\end{matrix}\; q;\, xy\right)$$

and

$$V(x) = {}_{G}\Phi_{H}\left(\begin{matrix}[(g : \zeta)]; \\ [(h : \rho)];\end{matrix}\; q;\, xz\right)$$

then in view of (1.1) and (1.2), we obtain

$$\sum_{n \geq 0} \frac{(-1)^{n}\, q^{n(n+1)/2}\, (q^{-\lambda} \;;\, q)_{n}\, q^{n(\gamma-1)}\, x^{n}\, z^{n} \prod_{i=1}^{G} (g_{i} \;;\, q)_{n\zeta_{i}}}{(q \;;\, q)_{n}\, (\gamma q^{-\lambda} \;;\, q)_{n} \prod_{i=1}^{H} (h_{i} \;;\, q)_{n\rho_{i}}}$$

$$\times \; {}_{A+1}\Phi_{B+1}\left(\begin{matrix}[(a : \theta)],\, \gamma \;; \\ ((b : \alpha)],\, \gamma q^{-n-\lambda} \;;\end{matrix}\; q;\, xyq^{n}\right)$$

$$\times \; {}_{G}\Phi_{H}\left(\begin{matrix}[(g)\, q^{n} : \zeta] \;; \\ [(h)\, q^{n} : \rho] \;;\end{matrix}\; q \;;\, xz\right)$$

$$= \Phi^{1\,:\,G\,;\,A}_{1\,:\,H\,;\,B}\left(\begin{matrix}\gamma : [(g : \zeta)] \;;\; [(a : \theta)] \;; \\ \gamma q^{-\lambda} : [(h : \rho)] \;;\; [(b : \alpha)] \;;\end{matrix}\; q \;;\, xz,\, xy\right) \qquad \text{...(3.3)}$$

valid for $|xy|$, $|xz|$, $|q| < 1$.

It is evident that right hand sides of (3.2) and (3.3) are identical and hence by an appeal to analytic continuation, we find that

$$\sum_{n \geq 0} \frac{(-1)^{n}\, q^{n(n+1)/2}\, (q^{-\lambda} \;;\, q)_{n}\, q^{n(\gamma-1)}\, x^{y}\, y^{n} \prod_{i=1}^{A} (a_{i} \;;\, q)_{n\theta_{i}}}{(q \;;\, q)_{n}\, (\gamma q^{-\lambda} \;;\, q)_{n} \prod_{i=1}^{B} (b_{i} \;;\, q)_{n\alpha_{i}}}$$

$$\times \; {}_{G+1}\Phi_{H+1}\left(\begin{matrix}[(g : \zeta)],\, \gamma \;; \\ [(h : \rho)],\, \gamma q^{n-\lambda} \;;\end{matrix}\; q;\, xzq^{n}\right)$$

$$\times\ {}_{A}\Phi_{B}\left(\begin{matrix}[(a)\ q^{n} : (\theta)]\ ; \\ [(b)\ q^{n} : \alpha]\ ;\end{matrix}\ q\ ;\ xy\right)$$

$$= \sum_{n\geq 0} \frac{(-1)^{n}\ q^{n(n+1)/2}\ (q^{-\lambda}\ ;\ q)_{n}\ q^{n(\gamma-1)}\ x^{n}\ z^{n} \prod_{i=1}^{A} (a_{i}\ ;\ q)_{n\theta_{i}}}{(q\ ;\ q)_{n}\ (\gamma q^{-\lambda}\ ;\ q)_{n} \prod_{i=1}^{B} (b_{i}\ ;\ q)_{n\alpha_{i}}}$$

$$\times\ {}_{G+1}\Phi_{H+1}\left(\begin{matrix}[(g : \zeta)],\ \gamma\ ; \\ [(h : \rho)],\ \gamma q^{n-\lambda}\ ;\end{matrix}\ q;\ xyq^{n}\right)$$

$$\times\ {}_{A}\Phi_{B}\left(\begin{matrix}[(a)\ q^{n} : \theta]\ ; \\ [(b)\ q^{n} : \alpha]\ ;\end{matrix}\ q\ ;\ xy\right)$$

$$= \sum_{n\geq 0} \frac{(-1)^{n}\ q^{n(n+1)/2}\ (q^{-\lambda}\ ;\ q)_{n}\ q^{n(\gamma-1)}\ x^{n}\ z^{n} \prod_{i=1}^{G} (g_{i}\ ;\ q)_{n\zeta_{i}}}{(q\ ;\ q)_{n}\ (\gamma q^{-\lambda}\ ;\ q)_{n} \prod_{i=1}^{H} (h_{i}\ ;\ q)_{n\rho_{i}}}$$

$$\times\ {}_{A+1}\Phi_{B+1}\left(\begin{matrix}[(a : \theta)],\ \gamma\ ; \\ ((b : \alpha)],\ \gamma q^{n-\lambda}\ ;\end{matrix}\ q;\ xyq^{n}\right)$$

$$\times\ {}_{G}\Phi_{H}\left(\begin{matrix}[(g)\ q^{n} : \zeta]\ ; \\ [(h)\ q^{n} : \rho]\ ;\end{matrix}\ q\ ;\ xz\right) \qquad \ldots(3.4)$$

whenever both sides exist.

Now, comparing the coefficients of

$$\frac{(-1)^{n}\ q^{n(n+1)/2}\ (q^{-\lambda}\ ;\ q)_{n}\ q^{n(\gamma-1)}\ (\gamma\ ;\ q)_{m}}{(q\ ;\ q)_{n}\ (\gamma q^{-\lambda}\ ;\ q)_{n+m}}$$

on both sides of (3.4), we get after some simplification,

$$\frac{\prod_{i=1}^{A} (a_{i},\ q)_{n\theta_{i}} \prod_{i=1}^{G} (g\ ;\ q)_{m\zeta_{i}}\ y^{n}\ z^{m}}{\prod_{i=1}^{B} (b_{i};\ q)_{n\alpha_{i}} \prod_{i=1}^{H} (h_{i} : q)_{m\rho_{i}}}\ {}_{A}\Phi_{B}\left(\begin{matrix}[(a)q^{n} : \theta]\ ; \\ [(b)q^{n} : \alpha]\ ;\end{matrix}\ q,\ xy\right)$$

$$= \frac{\prod_{i=1}^{G} (g_{i},\ q)_{n\zeta_{i}} \prod_{i=1}^{A} (a_{i}\ ;\ q)_{m\theta_{i}}\ z^{n}\ y^{m}}{\prod_{i=1}^{H} (h_{i};\ q)_{n\rho_{i}} \prod_{i=1}^{B} (b_{i} : q)_{m\alpha_{i}}}\ {}_{G}\Phi_{H}\left(\begin{matrix}[(g)q^{n} : \zeta]\ ; \\ [(h)q^{n} : \rho]\ ;\end{matrix}\ q,\ xz\right). \qquad \ldots(3.5)$$

Now, multiplying both sides of (3.5) by

$$\frac{\prod_{i=1}^{E}(e_i\,;\,p)_{m\nu_i'}\,\zeta^m}{\prod_{i=1}^{F}(f_i\,;\,p)_{m\epsilon_i}}$$

and summing with respect to m from $m = 0$ to $m = \infty$, it gives

$$\frac{\prod_{i=1}^{A}(a_i;\,q)_{n\theta_i}\,y^n}{\prod_{i=1}^{B}(b_i:q)_{n\alpha_i}}\;{}_A\Phi_B\left(\begin{matrix}[(a)q^n:\theta]\,;\\ [(b)q^n:\alpha]\,;\end{matrix}\;q;\,xy\right)$$

$$\times\;\Phi^{G:E}_{H:F}\left(\begin{matrix}q:[(g:\zeta)],\,p:[(e:\nu')]\,;\;z\xi\\ q:[(h:\rho)],\,p:[(f:\epsilon)]\,;\end{matrix}\right)$$

$$=\frac{\prod_{i=1}^{G}(g_i;\,q)_{n\zeta_i}\,z^n}{\prod_{i=1}^{H}(h_i\,;\,q)_{n\rho_i}}\;{}_G\Phi_H\left(\begin{matrix}[(g)q^n:\zeta]\,;\\ [(h)q^n:\rho]\,;\end{matrix}\;q;\,xz\right)$$

$$\times\;\Phi^{A:E}_{B:F}\left(\begin{matrix}q:[(a:\theta)],\,p:[(e:\nu')]\,;\;y\xi\\ q:[(b:\alpha)],\,p:[(f:\epsilon)]\,;\end{matrix}\right). \qquad \ldots(3.6)$$

Again comparing the coefficients of $x^R/(q;q)_R$ on both sides of (3.6), and then putting $n+R = m$, it yields

$$\frac{\prod_{i=1}^{A}(a_i;\,q)_{m\theta_i}\,y^m}{\prod_{i=1}^{B}(b_i:q)_{m\alpha_i}}\;\Phi^{G:E}_{H:F}\left(\begin{matrix}q:[(g:\zeta)],\,p:[(e:\nu')]\,;\;z\xi\\ q:[(h:\rho)],\,p:[(f:\epsilon)]\,;\end{matrix}\right)$$

$$=\frac{\prod_{i=1}^{G}(g_i;\,q)_{m\zeta_i}\,z^m}{\prod_{i=1}^{H}(h_i:q)_{m\rho_i}}\;\Phi^{A:E}_{B:F}\left(\begin{matrix}q:[(a:\theta)],\,p:[(e:\nu')]\,;\;y\xi\\ q:[(b:\alpha)],\,p:[(f:\epsilon)]\,;\end{matrix}\right) \qquad \ldots(3.7)$$

Now, multiplying both sides of (3.7) by

$$\frac{\prod_{i=1}^{C} (c_i\,;\,t)_{m\delta_i}\, n^m}{\prod_{i=1}^{D} (d_i\,;\,t)_{m\beta_i}}$$

and then summing both sides with respect to m from $m = 0$ to ∞, we finally arrive at the result (3.1).

It is evident that, following the above method, one can easily establish transformations involving multivariable basic hypergeometric functions of several variables of Srivastava, when the complex constants are each equal to unity. We mention below the following interesting transformation, without proof.

$$\Phi^{G:E'+C';\ldots;E^{(n)}+C^{(n)}}_{H:F'+D';\ldots;F^{(n)}+D^{(n)}}\left(\begin{matrix}(g) : (e'),\ (c');\ldots;(e^{(n)}),\ (c^{(n)})\ ; \\ (h) : (f'),\ (d');\ldots;(f^{(n)}),\ (d^{(n)})\ ;\end{matrix}\, z_1\,\xi_1,\ldots,z_n\,\xi_n\right)$$

$$\times\ \Phi^{L:A'+K';\ldots;A^{(n)}+K^{(n)}}_{P:B'+S';\ldots;B^{(n)}+S^{(n)}}\left(\begin{matrix}(l) : (a'),\ (k');\ldots;(a^{(n)}),\ (k^{(n)})\ ; \\ (p) : (b'),\ (s');\ldots;(b^{(n)}),\ (s^{(n)})\ ;\end{matrix}\, y_1\,\eta_1,\ldots,y_n\,\eta_n\right)$$

$$=\ \Phi^{G:A'+C';\ldots;A^{(n)}+C^{(n)}}_{H:B'+D';\ldots;B^{(n)}+D^{(n)}}\left(\begin{matrix}(g) : (a'),\ (c');\ldots;(a^{(n)}),\ (c^{(n)})\ ; \\ (h) : (b'),\ (d');\ldots;(b^{(n)}),\ (d^{(n)})\ ;\end{matrix}\, y_1\,\xi_1,\ldots,y_n\,\xi_n\right)$$

$$\times\ \Phi^{L:E'+K';\ldots;E^{(n)}+K^{(n)}}_{P:F'+S';\ldots;F^{(n)}+S^{(n)}}\left(\begin{matrix}(l) : (e'),\ (k');\ldots;(e^{(n)}),\ (k^{(n)})\ ; \\ (p) : (f'),\ (s');\ldots;(f^{(n)}),\ (s^{(n)})\ ;\end{matrix}\, z_1\,\eta_1,\ldots,z_n\,\eta_n\right) \qquad \ldots(3.8)$$

provided both sides exists and

$$|z_i\,\xi_i|,\ |y_i\,\eta_i|,\ |y_i\,\xi_i|,\ |z_i\eta_i| < 1$$

$(i = 1,\ldots,n)$, where

$$\Phi^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}(a) : (b');\ldots;(b^{(n)})\ ; \\ (c) : (d');\ldots;(d^{(n)})\ ;\end{matrix}\, q\ ;\ x_1,\ldots,x_n\right)$$

$$= \sum_{m_1,\ldots,m_n=0}^{\infty} \frac{\prod_{i=1}^{A} (a_i\,;\,q)_{m_1+\ldots+m_n} \prod_{i=1}^{B'} (b_i'\,;\,q)_{m_1} \cdots \prod_{i=1}^{B^{(n)}} (b_i^{(n)}\,;\,q)_{m_n}}{\prod_{i=1}^{C} (c_i\,;\,q)_{m_1+\ldots+m_n} \prod_{i=1}^{D'} (d_i'\,;\,q)_{m_1} \cdots \prod_{i=1}^{D^{(n)}} (d_i^{(n)}\,;\,q)_{m_n}}$$

$$\times\ \frac{x_1^{m_1}}{(q\,;\,q)_{m_1}} \cdots \frac{x_n^{m_n}}{(q : q)_{m_n}} \qquad \ldots(3.9)$$

Valid for $|x_i| < 1$ $(i = 1,\ldots,n)$, where (3.9) follows as a very special case of the basic analogue of generalized Lauricella hypergeometric function of n-variables due to Srivastava[9].

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