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HEAT CONDUCTION AND THE MULTIPLE HYPERGEOMETRIC FUNCTION OF SRIVASTAVA AND DAOUST

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In the present paper, we evaluate two integrals involving multiple hypergeometric function of Srivastava and Daoust (1969 a, b, 1972) and discuss their applications in solving a problem on heat conduction considered by Bhonsle (1966) and in establishing some expansion formulae involving the above function.

1. INTRODUCTION

Singh (1971) evaluated some integrals involving Kampé de Fériet function and one of them has been employed to obtain a solution of a problem in heat conduction given by Bhonsle (1966). Some expansion formulae involving above function have also been obtained.

In this paper, we evaluate two integrals involving the multiple hypergeometric function of Srivastava and Daoust (1969 a, b, 1972) defined by

$$S^{A\,:\,B';\,\ldots;\,B^{(n)}}_{C\,:\,D';\,\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;[d^{(n)}:\delta^{(n)}],\end{matrix}\; x_1,\ldots,x_n\right)$$

$$=\sum_{m_1,\ldots,m_n=0}^{\infty}\frac{\prod_{j=1}^{A}\Gamma(a_j+\sum_{i=1}^{n}m_i\theta_j^{(i)})\prod_{j=1}^{B'}(b_j'+m_1\Phi_j')\ldots\prod_{j=1}^{B^{(n)}}\Gamma(b_j^{(n)}+m_n\Phi_j^{(n)})}{\prod_{j=1}^{C}\Gamma(c_j+\sum_{i=1}^{n}m_i\Psi_j^{(i)})\prod_{j=1}^{D'}\Gamma(d_j'+m_1\delta_j^{i})\ldots\prod_{j=1}^{D^{(n)}}\Gamma(d_j^{(n)}+m_n\delta_j^{(n)}}$$

$$\times\frac{x_1^{m_1}}{m_1!}\cdots\frac{x_n^{m_n}}{m_n!} \qquad \ldots(1.1)$$

where

$$\theta_j^{(i)},\ j = 1, \ldots, A;\ \Phi_j^{(i)},\ j = 1, \ldots, B^{(i)};\ \Psi_j^{(i)},\ j = 1, \ldots, c;$$

$$\delta_j^{(i)},\ j = 1, \ldots, D^{(i)};\quad 1 \leqslant i \leqslant n$$

are real and positive and (a) is taken to abbreviate the sequence of A parameters $a_1, \ldots,; a_A$ $b^{(i)}$ abbreviates the sequence of $B^{(i)}$ parameters $b_1^{(i)}, \ldots, b_{B^{(i)}}^{(i)}$, $i = 1, \ldots, n$ with similar interpretation for (c) and $(d^{(i)})$, $i = 1, \ldots, n$; etc. and their applications will be made in solving a problem on heat conduction given by Bhonsle (1966) and in establishing some expansion formulae involving the above function.

2. Formulae Required

Multiplying both sides of the equation (Lebedev 1965, 4.16.1) by $e^{-Z^2} H_{2\nu}(z)$ and using orthogonality property of Hermite polynomials (1965), we have

$$\int_{-\infty}^{\infty} z^{2\rho}\, \rho^{-z^2} H_{2v}(Z)\, dz = \frac{\sqrt{\pi}\, 2^{2(\nu-\rho)}\, \Gamma(2\rho+1)}{\Gamma(\rho-\nu+1)},\ \rho = 0, 1, 2, \ldots, \quad \ldots(2.1)$$

which will be useful in our investigations.

Another formula required in our investigation is given in Erdélyi *et al.* (1954)

$$\int_{-1}^{1} (1-z^2)^{\rho-1} P_\nu^\mu(z)\, dz$$

$$= \frac{2^\mu \pi\, \Gamma\left(\rho+\frac{\mu}{2}\right)\Gamma\left(\rho-\frac{\mu}{2}\right)}{\Gamma\left(1+\rho+\frac{\nu}{2}\right)\Gamma\left(\rho-\frac{\nu}{2}\right)\Gamma\left(\frac{\mu}{2}+\frac{\nu}{2}+1\right)\Gamma\left(-\frac{\nu}{2}-\frac{\mu}{2}+\frac{1}{2}\right)} \quad \ldots(2.2)$$

$2\mathrm{Re}\,(\rho) > |\,\mathrm{Re}\,(\mu)\,|$.

3. Integrals

Making an appeal to (2.1), we obtain

$$\int_{-\infty}^{\infty} z^{2\rho} e^{-z^2} H_{2\nu}(z)\, S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]\,[(b'):\Phi'];\ldots;\\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;\end{matrix}\right.$$

(*equation continued on p.* 373)

$$\begin{matrix}[(b^{(n)}:\Phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1 z^{2\alpha_1}, \ldots, x_n z^{2\alpha_n} \Bigg) dz$$

$$= \sqrt{\pi}\; 2^{2(\nu-\rho)}\, S^{A+1\,:\,B';\,\ldots;\,B^{(n)}}_{C+1\,:\,D';\,\ldots;\,D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\ [2\rho+1:2\alpha_1,\ldots,2\alpha_n]:\\ [(c):\Psi',\ldots,\Psi^{(n)}],\ [1+\rho-\nu:\alpha_1,\ldots,\alpha_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; \frac{x_1}{2^{2\alpha_1}},\ldots,\frac{x_n}{2^{2\alpha_n}}\right) \qquad \ldots(3.1)$$

where

$$1-\alpha_i+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}>0$$

$i = 1, \ldots, n$; $\rho = 0, 1, 2, \ldots$ and $\alpha_1, \ldots, \alpha_n$ are real and positive.

Similarly, an appeal to (2.2) shows that

$$\int_{-1}^{1} (1-z^2)^{\rho-1} P_\nu^\mu(z)\, S^{A\,:\,B';\ldots;B^{(n)}}_{C\,:\,D';\ldots;D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;\\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}:\Phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1(1-Z^2)^{\alpha_1};\ldots,x_n(1-Z^2)^{\alpha_n}\right) dz$$

$$= \frac{\pi\, 2^\mu}{\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(\frac{1-\mu-\nu}{2}\right)}\, S^{A+2\,:\,B';\ldots;B^{(n)}}_{C+2\,:\,D';\ldots;D^{(n)}} \left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\Psi',\ldots,\Psi^{(n)}],\end{matrix}\right.$$

$$\begin{matrix}\left[\rho+\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n\right],\left[\rho-\frac{\mu}{2}:\alpha_1,\ldots,\alpha_n\right]:\left[(b'):\Phi'\right];\ldots;\left[(b^{(n)}):\Phi^{(n)}\right]\\ \left[\rho-\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n\right],\left[1+\rho+\frac{\nu}{2}:\alpha_1,\ldots,\alpha_n\right]:\left[(d'):\delta'\right];\ldots;\left[(d^{(n)}):\delta^{(n)}\right]\end{matrix}$$

$$\left. x_1,\ldots,x_n\right) \qquad \ldots(3.2)$$

provided that $2\,\mathrm{Re}(\rho) > |\,\mathrm{Re}(\mu)\,|$

$$1+\sum_{j=1}^{C}\Psi_j^{(i)}+\sum_{j=1}^{D^{(i)}}\delta_j^{(i)}-\sum_{j=1}^{A}\theta_j^{(i)}-\sum_{j=1}^{B^{(i)}}\Phi_j^{(i)}>0$$

$i = 1, \ldots, n$ and $\alpha_1, \ldots, \alpha_n$ are real and positive.

4. Applications to Heat Condition

Hermite polynomials have been utilized by Kampé de Fériet (1958–59) in solving a heat conduction equation. Bhonsle (1966) has also employed Hermite polynomials in solving the partial differential equation

$$\frac{\partial\Phi}{\partial t} = K\frac{\partial^2 \Phi}{\partial z^2} - K\Phi z^2 \qquad \text{...(4.1)}$$

where $\Phi(z, t)$ tends to zero for large values of t and when $|z| \to \infty$, this equation is related to the problem of heat conduction given by Churchill (1958)

$$\frac{\partial \Phi}{\partial t} = K\frac{\partial^2 \Phi}{\partial z^2} - h_1(\Phi - \Phi_0) \qquad \text{...(4.2)}$$

provided that $\Phi_0 = 0$ and $h_1 = Kz^2$.

The solution of (4.1) given by Bhonsle (1966) is

$$\Phi(Z, t) = \sum_{r=0}^{\infty} Q_r\, e^{-(1+2r)Kt - z^2/2} H_r(Z). \qquad \text{...(4.3)}$$

We shall consider the problem of determining a function $\Phi(z, t)$, where

$$\Phi(z, 0) = z^{2\rho} e^{-z^2} S^{A:B';\ldots;B^{(n)}}_{C:D';\ldots;D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;\\ [(c):\Psi',\ldots,\Psi^{(n)}]:(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\Phi^{(n)}],\\ [(d^{(n)}),:\delta^{(n)}],\end{matrix}\; x_1 z^{2^{\alpha_1}}, \ldots, x_n z^{2^{\alpha_n}}\right). \qquad \text{...(4.4)}$$

Now making an appeal to (4.3), (4.4) and the integral (3.1) with the orthogonality property of Hermite polynomials {See, for instance, Erdélyi *et al.* (1954, p. 289)}, we get the solution (4.3) of the problem in the form

$$\Phi(z, t) = \sum_{r=0}^{\infty} e^{-(1+2r)Kt - z^2/2} H_r(z)\; \frac{2^{r-2\rho-1/2}}{r!}$$

$$\times S^{A+1:B';\ldots;B^{(n)}}_{C+1:D';\ldots:D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\ [2\rho+1:2\alpha_1,\ldots,2\alpha_n]:\\ [(c):\Psi',\ldots,\Psi^{(n)}],\ [1+\rho-r/2:\alpha_1,\ldots,\alpha_n]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}],\\ [(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; \frac{x_1}{2^{2\alpha_1}},\ldots, \frac{x_n}{2^{2\alpha_n}}\right) \qquad \text{...(4.5)}$$

where all conditions of (3.1) are satisfied.

5. Expansion Formulae

An appeal to main integrals (3.1) with orthogonality property of Hermite polynomials (Erdélyi *et al.* 1954, p. 289) gives the expansion formulae

$$z^{2\rho}\, e^{-z^2/2}\, S^{A\,:\,B';\,\ldots;\,B^{(n)}}_{C\,:\,D';\,\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:[(b'):\Phi'];\ldots;\\ [(c):\Psi',\ldots,\Psi^{(n)}]:[(d'):\delta'];\ldots;\end{matrix}\right.$$

$$\left.\begin{matrix}[(b^{(n)}):\Phi^{(n)}];\\ [(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1\, z^{2\alpha_1},\ldots,x_n\, z^{2\alpha_n}\right)$$

$$=\sum_{r=0}^{\infty}\frac{2^{r-2\rho-1/2}}{r\,!}\, H_r(z)\, S^{A+1\,:\,B';\,\ldots;\,B^{(n)}}_{C+1\,:\,D';\,\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],\\ [(c):\Psi',\ldots,\Psi^{(n)}],\end{matrix}\right.$$

$$\left.\begin{matrix}[2\rho+1:2\alpha_1,\ldots,2\alpha_n]:[(b').\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [1+\rho-r/2:\alpha_1,\ldots,\alpha_n]:[(d'):\delta'];\ldots;[(d^{(n)}):\delta^{(n)}];\end{matrix}\; x_1/2^{2\alpha_1},\ldots,x_n/2^{2\alpha_n}\right). \qquad \ldots(5.1)$$

Similarly an appeal to (3.2) with the orthogonal property of associated Legendre function (Erdélyi *et al.* 1954, p. 278) shows that

$$(1-z^2)^{\rho-1}\, S^{A\,:\,B';\,\ldots;\,B^{(n)}}_{C\,:\,D';\,\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}]:\\ [(c):\Psi',\ldots,\Psi^{(n)}]:\end{matrix}\right.$$

$$\left.\begin{matrix}[(b'):\Phi'];\ldots;[(b^{(n)}):\Phi^{(n)}];\\ [(d'):\delta';\ldots;[(d^{(n)}:\delta^{(n)});\end{matrix}\; x_1(1-z^2)^{\alpha_1},\ldots,x_n(1-z^2)^{\alpha_n}\right)$$

$$=\sum_{\nu=0}^{\infty}\frac{\pi\, 2^{\mu-1}(2\nu+1)(\nu-\mu)\,!}{(\mu+\nu)\,!\,\Gamma\left(\frac{\mu+\nu}{2}+1\right)\Gamma\left(\frac{1-\mu-\nu}{2}\right)}\, P^{\mu}_{\nu}(z)$$

$$\times\, S^{A+2\,:\,B';\,\ldots;\,B^{(n)}}_{C+2\,:\,D';\,\ldots;\,D^{(n)}}\left(\begin{matrix}[(a):\theta',\ldots,\theta^{(n)}],[\rho+\mu/2:\alpha_1.\ldots,\alpha_n],\\ [(c):\Psi',\ldots,\Psi^{(n)}],[1+\rho+\nu/2:\alpha_1,\ldots,\alpha_n]\end{matrix}\right.$$

$$\left.\begin{matrix}[\rho-\mu/2:\alpha_1,\ldots,\alpha_n]:[(b'):\Phi'],\ldots,[(b^{(n)}):\Phi^{(n)}];\\ [\rho-\nu/2:\alpha_1,\ldots,\alpha_n]:[(d'):\delta'];\ldots,[d^{(n)}:\delta^{(n)};]\end{matrix}\; x_1;\ldots;x_n\right). \qquad \ldots(5.2)$$

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