THE DOUBLE-ENDED QUEUE WITH LIMITED WAITING SPACE*

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The double-ended queue involving taxis and customers at a taxi stand has been considered under the assumption that there is limited waiting space both for taxis and for customers. The arrival distribution of taxis is simulated by means of an arrival timing channel consisting of j phases. The arrivals of customers are assumed to be Poisson distributed. In the time dependent case we obtain an expression for the Laplace transform of the probabilities that there are n units in the queue. The cases of exponential and 2-Erlang arrival distributions for taxis have been considered as particular cases. In the steady state, for a 2-Erlang arrival distribution for taxis, we have evaluated the probabilities that the waiting space for customers or taxis is full in different particular cases. Their values are also shown by means of graphs.

INTRODUCTION

Kashyap (1965*a*) has discussed the double-ended queue involving queues of taxis and customers at a taxi stand with limited waiting space for both taxis and customers, and the arrivals of customers and taxis both being assumed to be according to the Poisson distribution. Kashyap (1965*b*) further extended the result to the case of general arrival distribution for the taxis by using the supplementary variable technique and succeded in finding the Laplace transform of the state probability generating function in the time dependent case. In the present paper we use the phase technique to study the same queueing process. This approach has the advantage of lending itself to easy computation of steady state probabilities in the special case of 2-Erlang arrival distribution of taxis which was not possible in the previous study (Kashyap 1965*b*). In the particular case of exponential arrival distribution for taxis the corresponding results of Kashyap (1965*a*), and

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Jaiswal (1961) are shown to follow. For a 2-Erlang arrival distribution, the probabilities that the waiting space for customers or taxis is full, are evaluated for different values of ρ, N and M, where $\rho = \dfrac{\lambda}{\mu}$ and N and M denote respectively the maximum number of customers and taxis allowed to wait at the stand.

Statement of the Problem

Customers arrive at a taxi stand according to a Poisson distribution with mean arrival rate λ, and depart with a taxi if they find a queue of taxis waiting, or else form a queue. The arrival distribution of taxis at the stand is simulated by means of an arrival timing channel consisting of j phases, each phase being exponential with mean $\dfrac{1}{\mu}$, the channel being fed by an infinite source. Phases are numbered in the reverse order. Each time the arrival channel becomes empty, a unit (taxi) is put into the channel, starting at the rth phase with probability C_r $(r = 1, 2, m..., j)$. After completing the rth phase, it goes to the $(r-1)$th phase, then to the $(r-2)$th and so on till the 1st phase, after finishing which it arrives at the stand leaving the channel empty. After arriving at the stand, the taxi departs taking one customers from the queue, if any, or waits.

There is limited waiting space for M taxis or N customers. Let $P_{n,r}(t)$ be the probability that at time t there are n customers waiting at the stand, the arriving taxi being in the rth phase. When n is positive, customers are waiting; when n is zero, neither taxis nor customers are waiting; and when n is negative, it signifies that $-n$ taxis are waiting.

Formulation of Equations

Following Keilson and Kooharian (1960), we have the following transition equations for the system :

$$P_{-M,r}(t+\Delta) = P_{-M,r}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta P_{-M,r+1}(t)+\mu\Delta C_r P_{-M+1,1}(t)$$
$$+\mu\Delta C_r P_{-M,1}(t), \; [1 \leqslant r < j] \quad \ldots \quad \ldots \quad \ldots \quad (1)$$

$$P_{-M,j}(t+\Delta) = P_{-M,j}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta C_j P_{-M+1,1}(t)+\mu\Delta C_j P_{-M,1}(t) \quad \ldots \quad (2)$$

$$P_{n,r}(t+\Delta) = P_{n,r}(t)[1-(\lambda+\mu)\Delta+\mu\Delta P_{n,r+1}(t)+\mu\Delta C_r P_{n+1,1}(t)$$
$$+\lambda\Delta P_{n-1,r}(t), \; [-M < n < N, \; 1 \leqslant r < j] \quad \ldots \quad (3)$$

$$P_{n,j}(t+\Delta) = P_{n,j}(t)[1-(\lambda+\mu)\Delta]+\mu\Delta C_j P_{n+1,1}(t)+\lambda\Delta P_{n-1,j}(t),$$
$$[-M < n < N] \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (4)$$

$$P_{N,r}(t+\Delta)=P_{N,r}(t)[1-\mu\Delta]+\mu\Delta P_{N,r+1}(t)+\lambda\Delta P_{N-1,r}(t), \ [1 \leqslant r < j] \quad \dots \quad (5)$$

$$P_{N,j}(t+\Delta) = P_{N,j}(t)[1-\mu\Delta]+\lambda\Delta P_{N-1,j}(t). \quad \dots \quad \dots \quad \dots \quad \dots \quad (6)$$

Transposing and letting $\Delta \to 0$ we have

$$\frac{d}{dt}P_{-M,r}(t)+(\lambda+\mu)P_{-M,r}(t) = \mu P_{-M,r+1}(t)+\mu C_r P_{-M+1,1}(t)+\mu C_r P_{-M,1}(t) \quad (7)$$

$$\frac{d}{dt}P_{-M,r}(t)+(\lambda+\mu)P_{-M,r}(t) = \mu C_j P_{-M+1,1}(t)+\mu C_j P_{-M,1}(t) \quad \dots \quad \dots \quad (8)$$

$$\frac{d}{dt}P_{n,r}(t)+(\lambda+\mu)P_{n,r}(t) = \mu P_{n,r+1}(t)+\mu C_r P_{n+1,1}(t)+\lambda P_{n-1,r}(t) \quad \dots \quad (9)$$

$$\frac{d}{dt}P_{n,j}(t)+(\lambda+\mu)P_{n,j}(t) = \mu C_j P_{n+1,1}(t)+\lambda P_{n-1,j}(t) \quad \dots \quad \dots \quad \dots \quad (10)$$

$$\frac{d}{dt}P_{N,r}(t)+\mu P_{N,r}(t) = \mu P_{N,r+1}(t)+\lambda P_{N-1,r}(t) \quad \dots \quad \dots \quad \dots \quad \dots \quad (11)$$

$$\frac{d}{dt}P_{N,j}(t)+\mu P_{N,j}(t) = \lambda P_{N-1,j}(t). \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad (12)$$

Now let us define the following generating functions

$$F_r(\alpha, t) = \sum_{n=-M}^{N} \alpha^n P_{n,r}(t) \quad \text{and} \quad F(\alpha, \beta, t) = \sum_{r=1}^{j} F_r(\alpha, t)\beta^r.$$

Multiplying (7) to (12) by appropriate powers of α and β and summing over n from $n = -M$, to $n = N$, and r from $r = 1$ to $r = j$, we have

$$\frac{d}{dt}F(\alpha, \beta, t)+\left[\lambda(1-\alpha)+\mu\left(1-\frac{1}{\beta}\right)\right]F(\alpha, \beta, t) = \mu F_1(\alpha, t)\left\{\frac{C(\beta)}{\alpha}-1\right\}$$

$$+\lambda a^N(1-\alpha)\sum_{r=1}^{j} P_{N,r}(t)\beta^r+\mu\alpha^{-M}P_{-M,1}(t)C(\beta)\left(1-\frac{1}{\alpha}\right) \quad \dots \quad (13)$$

where $C(\beta) = \sum_{r=1}^{j} \beta^r.C_r$.

Further we define $\overline{P}_{n,r}(s)$ to be Laplace transform (L.T.) of $P_{n,r}(t)$, viz.,

$$\overline{P}_{n,r}(s) = \int_0^\infty e^{-st}P_{n,r}(t)dt \quad \dots \quad \dots \quad \dots \quad (14)$$

the transforms of other functions being similarly denoted by corresponding letters under a bar, thus for example, $\overline{F}(\alpha, \beta, s)$ is the L.T. of $F(\alpha, \beta, t)$.

Let the system start with the arrival of a customer which makes the number of customers equal to i and let the taxi in the arrival channel be in the mth phase, so we have

$$P_{n,r}(o) = \delta_{i,n} \cdot \delta_{m,r} \qquad \ldots \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15)$$

where $\delta_{i,j} = 1, \quad i = j$

$= 0, \quad i \neq j$

therefore,

$$F(\alpha, \beta, o) = \alpha^i . \beta^m. \qquad \ldots \qquad \ldots \qquad \ldots \qquad (15')$$

Now taking L.T. of (13), we have

$$\left[s+\lambda(1-\alpha)+\mu\left(1-\frac{1}{\beta}\right)\right]\bar{F}(\alpha, \beta, s) = \alpha^i . \beta^m + \lambda\alpha^N(1-\alpha) \sum_{r=1}^{j} \bar{P}_{N,r}(s) . \beta^r$$

$$+\mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)\bar{P}_{-M,1}(s) . C(\beta) - \mu\bar{F}_1(\alpha, s)\left\{1-\frac{C(\beta)}{\alpha}\right\}. \qquad \ldots \qquad (16)$$

Since the generating functions $F_r(\alpha, t)$ and $F(\alpha, \beta, t)$ are finite sums, α and β may be taken as any complex numbers, thus choosing β such that the coefficient of $\bar{F}(\alpha, \beta, s)$ in the above expression is zero helps to evaluate $\bar{F}_1(\alpha, s)$.

Accordingly now putting $\beta = \dfrac{\mu}{[s+\lambda(1-\alpha)+\mu]} = Z$ (says),

(this amounts to choosing an appropriate β for a given α), we have

$$\bar{F}_1(\alpha, s) = \frac{\left[\alpha^i Z^m + \lambda\alpha^N(1-\alpha) \sum_{r=1}^{j} \bar{P}_{N,r}(s) Z^r + \mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right) C(Z) \bar{P}_{-M,1}(s)\right]}{\mu\left[1-\frac{1}{\alpha} . C(Z)\right]} \qquad (17)$$

where $C(Z) = \sum_{r=1}^{j} C_r . Z^r$.

The denominator is a polynomial of degree $(j+1)$ in α and therefore, has $j+1$ zeros. By Rouche's theorem, it can be seen that this equation has j roots inside and one outside the unit circle. Since the expression on the L.H.S. of (17) is regular over the entire complex plane the numerator must vanish at all the $j+1$ zeros of the denominator, giving rise to $j+1$ equations in the $j+1$ unknowns, viz.,

$$\bar{P}_{-M,1}(s) \text{ and } \bar{P}_{N,r}(s), \quad r = 1, 2, \ldots, j.$$

Putting $\beta = 1$ in eqn. (16) we get

$$\bar{F}(\alpha, 1, s) = \frac{\alpha^{i+1} + (1-\alpha)\left[\lambda\alpha^{N+1} \sum_{r=1}^{j} \bar{P}_{N,r}(s) - \mu\alpha^{-M} . \bar{P}_{-M,1}(s) + \mu\bar{F}_1(\alpha, s)\right]}{\alpha[s+\lambda(1-\alpha)]} \qquad (18)$$

It gives the generating function of the L.T. of the probabilities that there are n units present in the queue. Equation (17) and $(j+1)$ equations completely determine (18). We are justified here in taking value of $\overline{F}_1(\alpha, s)$ from (17), since it is independent of β.

Particular Cases.

(i) *Exponential Arrival Time Distribution*

In this case $C_r = \delta_{r,1}$, and $m = 1$, therefore, the zeros of the denominator are given by $\lambda\alpha^2-(s+\lambda+\mu)\alpha+\mu = 0$, which gives two distinct roots, let these be α_1, and α_2. So, now from eqn. (17) we have

$$\alpha_1^{-M} . \overline{P}_{-M,1}(s)-\frac{\lambda}{\mu}\alpha_1^{N+1} . \overline{P}_{N,1}(s)+\frac{\alpha_1^{i+1}}{\mu(\alpha_2-1)} = 0 \quad \ldots \quad (19)$$

$$\alpha_2^{-M} . \overline{P}_{-M,1}(s)-\frac{\lambda}{\mu}\alpha_2^{N+2} . \overline{P}_{N,1}(s)+\frac{\alpha_2^{i+1}}{\mu(\alpha_2-1)} = 0. \quad \ldots \quad (20)$$

Hence

$$\overline{P}_{N,1}(s) = \frac{[\alpha_1^{i+1} . \alpha_2^{-M}(1-\alpha_2)-\alpha_2^{i+1} . \alpha_1^{-M}(1-\alpha_1)]}{s[\alpha_1^{N+1} . \alpha_2^{-M}-\alpha_2^{N+1} . \alpha_1^{-M}]} \quad \ldots \quad \ldots \quad (21)$$

$$\frac{\mu}{\lambda} . \overline{P}_{-M,1}(s) = \frac{(\alpha_1 . \alpha_2)^{i+1}[(\alpha_2^{N-i+1}-\alpha_1^{N-i+1})-(\alpha_2^{N-i}-\alpha_1^{N-i})]}{s[\alpha_2^{N+1} . \alpha_1^{-M}-\alpha_1^{N+1} . \alpha_2^{-M}]}. \quad \ldots \quad (22)$$

Eqns. (21) and (22) agree with corresponding results of Kashyap (1965*a*, p. 532) and in the particular case $M = 0$, these reduces to similar results of Jaiswal (1961, p. 113).

(ii) *2-Erlang Arrival Time Distribution*

In this case $C_r = \delta_{r,2}$ and $m = 2$, so we have from (17)

$$\alpha_1^{N+1}\left[\frac{1}{Z_1}\overline{P}_{N,1}(s)+\overline{P}_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_1^{-M} . \overline{P}_{-M,1}(s)+\frac{\alpha_1^{i+1}}{\lambda(1-\alpha_1)} = 0 \quad \ldots \quad (23)$$

$$\alpha_2^{N+1}\left[\frac{1}{Z_2}\overline{P}_{N,1}(s)+\overline{P}_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_2^{-M}\overline{P}_{-M,1}(s)+\frac{\alpha_2^{i+1}}{\lambda(1-\alpha_2)} = 0 \quad \ldots \quad (24)$$

$$\alpha_3^{N+1}\left[\frac{1}{Z_3}\overline{P}_{N,1}(s)+P_{N,2}(s)\right]-\frac{\mu}{\lambda}\alpha_2^{-M} . P_{-M,1}(s)+\frac{\alpha_3^{i+1}}{\lambda(1-\alpha_3)} = 0 \quad \ldots \quad (25)$$

where α_r, $r = 1, 2, 3$ are the root of

$$1-\frac{1}{\alpha}\left[\frac{\mu}{s+\lambda(1-\alpha)+\mu}\right]^2 = 0.$$

On solving (23), (24) and (25) we have

$$\bar{P}_{N,1}(s) = \frac{\Sigma \dfrac{\alpha_1^{i+1}(\alpha_2 \cdot \alpha_3)^{N+1}}{\lambda(1-\alpha_1)} \cdot [\alpha_3^{-(M+N+1)} - \alpha_2^{-(M+N+1)}]}{(\alpha_1 \cdot \alpha_2 \cdot \alpha_3)^{N+1} \, \Sigma \, \alpha_1^{-(M+N+1)} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]} \quad \ldots \quad (26)$$

$$\frac{\mu}{\lambda} \cdot P_{-M,1}(s) = \frac{\Sigma \dfrac{\alpha_1^{i-N}}{\lambda(1-\alpha_1)} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]}{\Sigma \, \alpha_1^{-(M+N+1} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]} \quad \ldots \quad \ldots \quad \ldots \quad (27)$$

$$\bar{P}_{N,2}(s) = \frac{\alpha_1^{-(M+N+1)} \cdot \Sigma \dfrac{\alpha_1^{i-X}}{\lambda(1-\alpha)} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]}{\Sigma \, \alpha_1^{-(M+N+1)} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]} - \frac{\alpha_1^{i-N}}{\lambda(1-\alpha_i)}$$

$$-\frac{1}{Z_1} \cdot \frac{\Sigma \dfrac{\Sigma \, \alpha_1^{i+1}(\alpha_2 - \alpha_3)^{N+1}}{\lambda(1-\alpha_1)} [\alpha_3^{-(M+N+1)} - \alpha_2^{-(M+N+1)}]}{(\alpha_1 \cdot \alpha_2 \cdot \alpha_3)^{N+1} \, \Sigma \, \alpha_1^{-(M+N+1)} \left[\dfrac{1}{Z_3} - \dfrac{1}{Z_2}\right]} \quad \ldots \quad (28)$$

where summation is carried over the roots $\alpha_1, \alpha_2, \alpha_3$ and Z_1, Z_2, Z_3, where

$$Z_i = \frac{\mu}{[s+\lambda(1-\alpha_i)+\mu]}, \quad i = 1, 2, 3.$$

Hence in this case

$$\bar{F}_i(\alpha, 1, s) = \frac{1}{\alpha[s+\lambda(1-\alpha)]} \left[\alpha^{i+1} + (1-\alpha) \left\{ \lambda \alpha^{N+1} \sum_{r=1}^{2} \bar{P}_{N,r}(s) - \mu \alpha^{-M} \cdot \bar{P}_{-M,1}(s) \right. \right.$$

$$\left. \left. + \frac{\alpha^i Z^2 + \lambda \alpha^N (1-\alpha) \sum_{r=1}^{2} \bar{P}_{N,r}(s) Z^r + \mu \alpha^{-M} \left(1 - \dfrac{1}{\alpha}\right) Z^2 \bar{P}_{-M,1}(s)}{\left[1 - \dfrac{Z^2}{\alpha}\right]} \right\} \right] \quad (29)$$

where $\bar{P}_{N,1}(s)$, $\bar{P}_{N,2}(s)$ and $\bar{P}_{-M,1}(s)$ are given by (26), (28) and (27).

Steady State Solution

For the steady state case we have the well known property.

$$\lim_{s \to 0} sP_{n,r}(s) = \lim_{t \to \infty} P_{n,r}(t) = P_{n,r}. \quad \ldots \quad \ldots \quad \ldots \quad (30)$$

Now applying (30) to (17) and (18) we have

$$F_1(\alpha) = \sum_{n=-M}^{N} \alpha^n . P_{n,1}$$

$$= \frac{\lambda\alpha^N(1-\alpha) \sum_{r=1}^{j} P_{N,r}\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r + \mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)P_{-M,1} \sum_{r=1}^{j} C_r\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r}{\mu\left[1-\frac{1}{\alpha} \sum_{r=1}^{j} C_r\left[\frac{\mu}{\lambda(1-\alpha)+\mu}\right]^r\right]} \quad \ldots \quad (31)$$

and

$$F(\alpha, 1) = \left[\alpha^N \sum_{r=1}^{j} P_{N,r} - \frac{\mu}{\lambda} \alpha^{-(M+1)} P_{-M,1} + \frac{\mu}{\lambda\alpha} F_1(\alpha)\right]. \quad \ldots \quad (32)$$

Now we restrict our discussion for the 2-Erlang arrival time distribution, for which we have from (31) and (32)

$$F_1(\alpha) = \sum_{n=-M}^{N} P_{n,1}\, \alpha^n$$

$$= \frac{\left[\lambda\alpha^N(1-\alpha) \sum_{r=1}^{2} P_{N,r}\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^r + \mu\alpha^{-M}\left(1-\frac{1}{\alpha}\right)\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^2 . P_{-M,1}\right]}{\mu\left[1-\frac{1}{\alpha}\left(\frac{\mu}{\mu+\lambda(1-\alpha)}\right)^2\right]} \quad \ldots \quad (33)$$

and

$$F(\alpha, 1) = \left[\alpha^N \sum_{r=1}^{2} P_{N,r} - \frac{\mu}{\lambda}\alpha^{-(M+1)} P_{-M,1} + \frac{\mu}{\lambda\alpha} F_1(a)\right] \quad \ldots \quad (34)$$

where $F_1(\alpha)$ is given by (33) and $P_{N,1}$, $P_{N,2}$ and $P_{-N,1}$ are to be determined by making $s \to 0$ in eqns. (26), (28), (27). We notice that now α_1, α_2, α_3 are the roots of

$$1-\frac{1}{\alpha}\left[\frac{\mu}{\mu+\lambda(1-\alpha)}\right]^2 = 0.$$

Here it is to be noted that $\alpha = 1$ (say α_1) is a root of the above equation, then α_2 and α_3 are the roots of

$$\alpha^2-\left(1+\frac{2}{\rho}\right)\alpha+\frac{1}{\rho^2} = 0$$

where $\rho = \lambda/\mu$. Thus now solving (26), (27) and (28)

$$P_{-M,1} = \frac{1}{2} \frac{(1-2/\rho)[\alpha_2-\alpha_3]}{\Sigma\, \alpha_1^{-(M+N+1)}[\alpha_2-\alpha_3]} \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad (35)$$

$$P_{N,1} = \frac{1}{2} \frac{\frac{(1-2\rho)}{\rho^2}\, [\alpha_2^{-(M+N+1)} - \alpha_3^{-(M+N+1)}]}{\Sigma\, \alpha_1^{-(M+Y+1)}\, [\alpha_2-\alpha_3]} \quad \ldots \quad \ldots \quad (36)$$

$$P_{N,2} = \left[\left\{\frac{1}{\rho}\, P_{-M,1}+1\right\}-\left\{P_{N,1}+\frac{1}{2\rho}\right\}\right]. \qquad \ldots \qquad \ldots \qquad (37)$$

It may be noted that $P_{-M,1}$ and $P_{N,1}$ are the probabilities that the waiting space respectively for taxis and customers is full when a taxi is about to arrive at the taxi stand.

Also $P_{N,1}+P_{N,2} = P_N$ gives the probability that the waiting space for customers is full irrespective of the phase of the arrival channel.

Numerical Results

In this section, we tabulate the values of $P_{N,1}$, $P_{N,2}$, $P_{-M,1}$ for different values of ρ, N and M $(= 5)$ for the 2-Erlang arrival distribution for taxis (Table I). These values are also shown in Figs. 1 and 2.

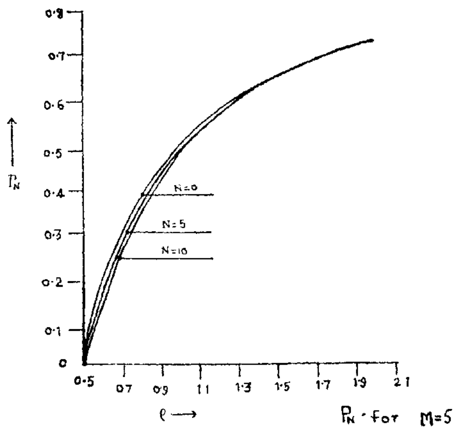

Fig. 1.

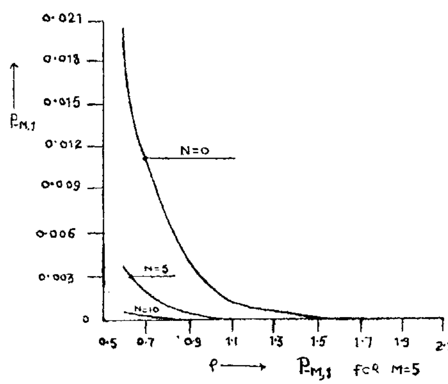

Fig. 2.

Table I

ρ \ N	$P_{-M,1}$ 0	$P_{-M,1}$ 5	$P_{-M,1}$ 10	$P_{N,1}$ 0	$P_{N,1}$ 5	$P_{N,1}$ 10	$P_{N,2}$ 0	$P_{N,2}$ 5	$P_{N,2}$ 10
0·5	0	0	0	0	0	0	0	0	0
0·6	0·02013	0·003605	0·00015	0·1339	0·1147	0·1112	0·069295	0·057809	0·05555
0·7	0·01621	0·001545	0·000012	0·1984	0·1848	0·1834	0·110267	0·102907	0·10211
0·8	0·0079	0·000273	0·000001	0·2436	0·2377	0·2374	0·141275	0·13764	0·137601
0·9	0·00408	0·000058	0·0000001	0·2806	0·2778	0·27771	0·16843	0·16729	0·16674
1·0	0·00225	0·000016	0·0000000	0·3101	0·30903	0·30902	0·19215	0·19098	0·19058
1·1	0·00120	0·000005	0·0000000	0·3347	0·3386	0·33380	0·21184	0·21159	0·21126
1·2	0·00065	0·000001	0·0000000	0·3540	0·3538	0·35371	0·22998	0·22955	0·22954
1·3	0·00031	0·000000	0·0000000	0·37290	0·3728	0·37273	0·24273	0·24260	0·24260
1·4	0·000014	0·000000	0·0000000	0·37946	0·3793	0·37918	0·26340	0·26355	0·26367
1·5	0·000006	0·000000	0·0000000	0·39509	0·39507	0·39506	0·271564	0·27158	0·27159
1·6	0·000000	0·000000	0·000000	0·4056	0·4052	0·4049	0·2819	0·2823	0·2826
1·7	0·000000	0·000000	0·000000	0·4138	0·4135	0·41340	0·2921	0·2924	0·2925
1·8	0·000000	0·000000	0·000000	0·4164	0·4162	0·4160	0·3058	0·3060	0·30620
1·9	0·000000	0·000000	0·000000	0·42726	0·42724	0·42722	0·30959	0·30961	0·30963
2·0	0·000000	0·000000	0·000000	0·43320	0·43310	0·43302	0·31680	0·31690	0·316980

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